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The private memory of aggregate uncertainty ☆

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ABSTRACT

We investigate social insurance in a dynamic Mirrlees's (1971) economy for which each agent's labor market productivity is the product of her stochastic and privately observed ability and an aggregate, publicly observed, stochastic component. The interaction between aggregate and idiosyncratic shocks optimally induces memory of aggregate uncertainty. We show that the optimal allocation depends on previous aggregate shocks when: *i*) preferences are not logarithmic; *ii*) capital accumulation is possible, or; *iii*) private type distributions depend on the aggregate state.

Using a representative agent endowment economy, Lucas and Stokey (1983) asked how taxes should optimally be spread across time and states of nature under the assumption that policy instruments are exogenously restricted. Werning (2007) expanded our understanding of the topic by adding heterogeneity and redistributive concerns and by considering unrestricted tax instruments. In this paper, we add yet another dimension of policy to the problem: social insurance.

We consider a world where shocks to labor productivity on an endowment economy drive the business cycles and focus on the non-stationary nature of allocations induced by incentive provision in an otherwise stationary economy. More precisely, we write a multi-period model that explores the trade-off between insurance and incentives, as in Golosov et al. (2003); Farhi and Werning (2013); Golosov et al. (2016), while incorporating the existence of aggregate shocks in a closed economy. Individuals with identical preferences face shocks to their productivity, but social insurance is hindered by informational asymmetry.[1] The informational frictions preclude the attainment of full insurance and generate incentive problems that affect the value of aggregate resources. By incorporating aggregate shocks to the model, the feedback between incentive provision and aggregate risk sharing is made apparent. Because policy instruments are only restricted by the

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[1] These models inspired by Mirrlees's (1971) original contribution are, of course, a small part of a larger literature of dynamic insurance that dates back to Townsend (1982); Rogerson (1985); Green (1987); Spear and Srivastava (1987).

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underlying informational structure of the problem, one simultaneously addresses the insurance provision problem itself and its financing in a coherent framework.

Marginal distortions in economies with private information are characterized by wedges: the gap in the marginal rates of substitution and transformation between consumption and labor. We find that, unless it is optimal for labor wedges to be invariant to aggregate states, which only occurs in knife-edge cases, allocations must display memory of aggregate shocks. In a static setting, wedges are introduced in an incentive problem to discourage shirking. The more redistribution we want to implement, the higher the wedges. Variation in wedges are, therefore, associated with different levels of redistribution across aggregate states. In a dynamic economy, incentives for labor provision in a certain period are adjusted via wedges within the period and promises of future utility.

Memory is introduced in the form of continuation utilities that depend on the current aggregate shock. It therefore makes future inequality depend on the current aggregate state. Since aggregate shocks are i.i.d. and independent of idiosyncratic shocks, in our baseline economy, it is not clear why one would want inequality to display this type of history dependence.[2] What this preliminary finding says is that, if it is desirable for redistribution to vary across states, then it will also be optimal for different levels of inequality to be inherited from different states.

If the utility of consumption is logarithmic, we show that wedges should be invariant to the aggregate state and future allocations should display no memory of aggregate uncertainty.[3] Neither current incentives nor future inequality should optimally depend on the current state of the economy. How special is this result, or, alternatively, how pervasive is memory? Unfortunately, fully characterizing the solution for the planner's problem is seldom possible.

In cases for which a partial characterization is possible, e.g., with iso-elastic preferences, then, with the exception of the logarithmic case we have mentioned, efficient allocations display memory of aggregate shocks.[4] For general separable preferences, we show that logarithmic consumption preferences are the only ones that guarantee the optimality of memoryless allocations: for any non-logarithmic preferences, it is possible to find a type distribution for which the optimal allocation has memory (Proposition 4). Note that, except in the case of logarithmic utility of consumption, the social marginal value of aggregate consumption is affected by the provision of incentive — see Demange (2008). In all other cases, current distortions are traded-off against future inequality in a state-dependent way.

In our baseline model, memory arises solely due to the need to generate incentives for an efficient provision of social insurance in an endowment economy. Accumulation of capital and state varying distribution of risks may add extra reasons for conditioning future allocations on current aggregate shocks. We explore these issues in the last section of the paper. We show that capital accumulation or aggregate state-varying type distribution lead to the presence of memory even in the logarithmic preferences case.

The rest of the paper is organized as follows. Section 1 describes the economy. Definitions of memory and incentive-feasible allocations are introduced in Section 2, where our first results are also proved. The main results in the paper are in Section 3. In Section 4 one finds the cases of capital accumulation and non-independence between aggregate shocks and distribution of types. Section 5 concludes. Longer proofs are collected in the appendix.

1. Environment

The economy is inhabited by a continuum of measure one of ex-ante identical individuals each living for T periods. To keep the discussion brief, we focus on $T < \infty$.

Every period individuals realize a temporary shock, $\theta_t \in \Theta = \{\theta_{(1)}, \ldots, \theta_{(N)}\}$, which affects the cost of supplying a given amount of effort, $n \in \mathbb{R}_+$. We use $\theta^t = (\theta_1, \ldots, \theta_t)$ to denote a history of idiosyncratic shocks up to period t. For any $\theta^{t+\tau}$ and θ^t, we say that $\theta^{t+\tau}$ is a continuation of history θ^t, denoted $\theta^{t+\tau} \succ \theta^t$, if the first t entries of $\theta^{t+\tau}$ are equal to θ^t.

For a given sequence of productivity shocks θ^T, agents have preferences defined over sequences of consumption–effort bundles, $(c, n) \equiv \{c_t, n_t\}_{t=1}^T$, represented by an inter- and intra-temporal additive separable utility function,

$$U(c,n) = \sum_{t=1}^{T} \beta^t \{\varphi(c_t) - \theta_t \zeta(n_t)\}, \tag{1}$$

where $c_t \in \mathbb{R}_+$ is period t consumption and $n_t \in [0, \bar{n}]$ is period t effort. $\varphi : \mathbb{R}_{++} \mapsto \mathbb{R}$ and $\zeta : \mathbb{R}_+ \mapsto \mathbb{R}$ are smooth, strictly increasing functions with both φ and $-\zeta$, strictly concave, $\lim_{c \searrow 0} \varphi'(c) = \infty$ and $\zeta'(0) = \zeta(0) = 0$.

Idiosyncratic shocks are not, however, the only source of uncertainty in this economy. Each period, an aggregate shock represented by a random variable $z_t \in Z$, Z a finite set, affects the economy's technology. $z^t \equiv (z_1, \ldots, z_t)$ denotes the history of aggregate shocks up to period t.

We use $f_t(\theta^t)$ to denote the probability that history θ^t is realized, and $f_{t+s}(\theta^{t+s}|\theta^t)$ the probability that history realizes θ^{t+s} given that, at t, history θ^t has realized. Of course, $f_{t+s}(\theta^{t+s}|\theta^t) = 0$ if θ^{t+s} is not a continuation of θ^t. Finally, we

[2] Werning (2007) finds no memory to be optimal in a model with redistributive but not insurance concerns, whereas Phelan (1994) makes the aggregate state informative about agents' effort in a moral hazard setting to generate memory.

[3] The result is naturally related to the findings in Atkeson and Lucas (1992).

[4] Partial characterization is also possible with slightly more general utility of consumption than iso-elastic. Memory characterize constrained efficient allocations in these cases too.

assume that the law of large numbers applies so that, at period t, state z^t, the cross-sectional distribution of types coincides with the distribution associated with f_t. We assume that aggregate shocks are i.i.d. and distributed according to $\Pi(z)$, with associated density $\pi(z)$. With some abuse in notation, $\Pi_t(z^t)$ is the product measure induced on Z^t. For notation brevity, we define $\pi_{t+s}(z^{t+s}|z^t)$ as $\Pi_{r=1}^{s}\pi(z_{t+r})$. Individual shocks $\{\theta_t\}_{t\geq 0}$ are independent from one another and from z^T.

We are now in a position to define feasible allocations. It will however be more convenient to work directly in the space of flow utilities. Let C be the inverse mapping of φ, which domain is the open interval I_φ.[5] Similarly, define ζ's inverse mapping N, which domain is $I_\zeta = [0, \infty)$. Then, for each (φ, ζ), let U be the set of mappings $\boldsymbol{u} = \{\boldsymbol{u}_t\}_{t=1}^{T}$ with $\boldsymbol{u}_t : \Theta^t \times Z^t \mapsto I_\phi$, H, the set of mappings $\boldsymbol{h} = \{\boldsymbol{h}_t\}_{t=1}^{T}$ with $\boldsymbol{h} : \Theta^t \times Z^t \mapsto I_\zeta$.

Noting that $\boldsymbol{u}(\theta^t, z^t) = \varphi(\boldsymbol{c}_t(\theta^t, z^t))$, $\boldsymbol{h}(\theta^t, z^t) = \zeta(\boldsymbol{n}_t(\theta^t, z^t))$, define an *allocation* as $(\boldsymbol{u}, \boldsymbol{h}) \equiv \{(\boldsymbol{u}_t, \boldsymbol{h}_t)\}_{t=1}^{T}$, with $(\boldsymbol{u}_t, \boldsymbol{h}_t) : Z^t \times \Theta^t \to U \times H$ for each t. Functions $\boldsymbol{u}_t(\theta^t, z^t)$ and $\boldsymbol{h}_t(\theta^t, z^t)$ are $(\theta^t, z^t)-$measurable. They denote the bundle of flow utility of consumption and dis-utility of effort allocated to an agent with history θ^t at period t, when aggregate history is z^t.

An allocation is feasible if, for every t, z^t,[6]

$$\sum_{\theta^t} f_t(\theta^t)\left[C(\boldsymbol{u}(\theta^t, z^t)) - z_t N(\boldsymbol{h}(\theta^t, z^t))\right] \leq 0. \tag{2}$$

The informational structure of the problem is such that idiosyncratic shocks are private information while aggregate shocks are publicly observed.

2. Constrained efficient allocations

A benevolent planner chooses a feasible allocation to maximize

$$\sum_t \beta^{t-1} \sum_{z^t} \pi_t(z^t) \sum_{\theta^t} f_t(\theta^t)\left[\boldsymbol{u}_t(\theta^t, z^t) - \theta_t \boldsymbol{h}_t(\theta^t, z^t)\right]. \tag{3}$$

From a period one's perspective this is equivalent to assuming that the government maximizes a utilitarian social welfare function.[7]

To pursue its goal the planner is restricted by the informational structure of the problem. It must then rely on a mechanism or game form to implement the allocation which maximizes its objective.

We assume that the planner is endowed with a commitment technology which allows us to rely on the revelation principle to characterize the set of implementable allocations using a truthful direct mechanism. Each period and state of the world the planner asks the agent his or her current shock, θ. The planner has perfect recall of past announcements, so reporting the complete history or reporting only the current shock is equivalent. Upon receiving a report from the agent the planner assigns her a pair $(\boldsymbol{u}, \boldsymbol{h})$ potentially dependent on the whole history of aggregate shocks and private reports.

Define a *reporting strategy*, $\sigma = \{\sigma_t\}_{t=1}^{T}$, as a sequence of mappings $\sigma_t : \Theta^t \times Z^t \to \Theta$, which associate to every history (θ^t, z^t) an announcement $\hat{\theta}$. Let $\sigma^t \equiv (\sigma_1, \ldots, \sigma_t)$. Because individuals cannot lie about the aggregate state of the economy, reports are restricted to the idiosyncratic shocks. Announcement strategies may, nonetheless, depend on z^t. Any reporting strategy σ, such that a possible history (θ^t, z^t) generates an inconsistent reported history $\sigma^t(\theta^t, z^t)$ is *not admissible*.[8] The set of all admissible strategies is represented by Σ.

Let $\sigma_t^*(\theta^t, z^t) = \theta^t$ for all t, θ^t, z^t be the truth-telling strategy. The incentive compatibility constraints are

$$\begin{aligned}
&\sum_t \beta^{t-1} \sum_{z^t} \pi_t(z^t) \sum_{\theta^t} f_t(\theta^t)\left[\boldsymbol{u}_t(\theta^t, z^t) - \theta_t \boldsymbol{h}_t(\theta^t, z^t)\right] \\
&\quad \geq \sum_t \beta^{t-1} \sum_{z^t} \pi_t(z^t) \sum_{\theta^t} f_t(\theta^t)\left[\boldsymbol{u}_t(\sigma(\theta^t, z^t), z^t) - \theta_t \boldsymbol{h}_t(\sigma(\theta^t, z^t), z^t)\right]
\end{aligned} \tag{4}$$

for all $\sigma \in \Sigma$.

An allocation that satisfies (4) and (2) is *incentive-feasible.*

The planner's problem is to choose among the incentive feasible allocations the one which maximizes the ex-ante expected utility of individuals. That is, its problem, which we call Program $\mathcal{P}_0$, is to maximize (3), subject to (2), and (4).

The transformation of variables we adopted guarantees that Program $\mathcal{P}_0$ is concave. Moreover, the allocation $(\boldsymbol{u}(\theta^t, z^{t-1}, z), \boldsymbol{h}(\theta^t, z^{t-1}, z)) = (\bar{\boldsymbol{u}}(z), \bar{\boldsymbol{h}}(z))$, for all t, θ^t, z^{t-1}, z and such that for all z, $C(\bar{\boldsymbol{u}}(z)) < zN(\bar{\boldsymbol{h}}(z))$ satisfies (4) and

[5] If $\inf I_\varphi > -\infty$, with abuse of notation we assume the point $\inf I_\varphi$ is included in I_φ and satisfies $C(\inf I_\varphi) = 0$.

[6] Capital accumulation may be added to the model. We refrain from doing so, since capital accumulation itself usually creates a form of history dependence which we want to isolate from the ones we discuss here. See, however, Section 4.2 for a brief discussion.

[7] Allowing for ex-ante heterogeneity or different Pareto weights increases notional burden and does not change the results in the paper.

[8] Formally, σ is admissible if, for all $t \geq 0$, $\sum_{(\theta^t, z^t) \in A_\sigma} \pi(z^t)\mu_t(\theta^t) = 1$, where $A_\sigma \equiv \{(\theta^t, z^t) \mid \sigma^t(\theta^t, z^t) \in supp(\mu_t)\}$. When shocks are perfectly persistent, this rules out announcing a different productivity in two different periods.

is feasible. Hence, if $(\boldsymbol{u}^\star, \boldsymbol{h}^\star)$ solves Program $\mathcal{P}_0$, there are Lagrange multipliers $(\mu^\star, \lambda^\star) := (\mu^\star(\sigma))_\sigma, (\lambda^\star(z^t))_{z^t,t}$ associated with constraints (2) and (4) such that $((\boldsymbol{u}^\star, \boldsymbol{h}^\star), (\mu^\star, \lambda^\star))$is a saddle point for the Lagrangian associated with program $\mathcal{P}_0$.

2.1. Memory

From Townsend (1982); Rogerson (1985); Green (1987); Spear and Srivastava (1987) we learn that efficient allocations will typically depend on past individual history. The question we ask is whether they should optimally depend on aggregate history as well.

Definition 1. If an allocation is such that, it is independent of z^{t-s} for all t, s, z^t, i.e., it is of the form

$$(\boldsymbol{u}_t(\theta^t, z^t), \boldsymbol{h}_t(\theta^t, z^t)) = (\tilde{\boldsymbol{u}}_t(\theta^t, z_t), \tilde{\boldsymbol{h}}_t(\theta^t, z_t))$$

then we say it **does not display memory**.

We start by examining the case $\varphi(c) = \ln(c)$. These preferences play an important role in macroeconomics for they are among a small class of preferences which are compatible with the empirical observation that labor supply has remained relatively constant despite the very large increase in labor productivity in the last two centuries.

Proposition 1. *Let $\varphi(c) = \ln(c)$ then the optimal allocation does not display memory of aggregate shocks.*

Proposition 1 is, of course, reminiscent of Atkeson and Lucas (1992). There, in a world without uncertainty, it is shown that a constant price equal to the discount rate supports the constrained efficient allocation. Here, the fact that incentive provision does not affect the marginal value of resources is used to show that efficient allocations take a very simple form even in the presence of aggregate uncertainty.

For this important class of preferences the optimal allocation does not display memory, independently of the process we consider for the idiosyncratic shocks. In what follows we assess whether this is a typical feature of constrained efficient allocations. A first, intermediate result – Lemma 1 – relates lack of memory to a form of separability in allocations that has important consequences for the behavior of prices and wedges – Proposition 2.

Lemma 1. *If an efficient allocation is such that the period t flow utility from consumption is of the form $\boldsymbol{u}_t(\theta^t, z^t) = \bar{\boldsymbol{u}}_t(\theta^t, z_t)$, then, for every $s < t$ there are functions $\varphi_s^u(\cdot)$ and $\eta_s(\cdot)$ such that, $C'(\boldsymbol{u}_s(\theta^s, z^s)) = \varphi_s^u(\theta^s)\eta_s(z_s)$, $\forall\, \theta^s, z^s$.*

3. Memory

In all that follows we focus on a restricted class of environments for which a recursive representation for the planner's program is possible. Toward this end we assume that the idiosyncratic shocks follow a first order Markov process. For every i, j, t, θ^{t-2}, we then write $f(i|j)$ to denote $f_t(\theta^{t-2}, \theta_{(j)}, \theta_{(i)}|\theta^{t-1}, \theta_{(j)})$

Section 3.1 considers a cost minimization problem which the allocation that solves $\mathcal{P}$ must also solve. We explore its recursive structure to characterize this allocation under a restricted class of preferences. We then show that, for any Markov process for the idiosyncratic shocks and any i.i.d. process for the aggregate shocks, the efficient allocations must display memory of aggregate uncertainty. Then in Section 3.2 we show that, unless the utility from consumption is of the form $\varphi(c) = \ln c$, it is always possible to find an economy for which the allocation must display memory.

3.1. The recursive program

Let

$$\boldsymbol{w}(\theta^t, z^t) = \sum_{s=1}^{T-t} \sum_{z^{t+s} \succ z^t} \pi_{t+s}(z^{t+s}|z^t) \sum_{\theta^{t+s} \succ \theta^t} f_{t+s}(\theta^{t+s}|\theta^t) \left[\boldsymbol{u}(\theta^{t+s}, z^{t+s}) - \theta_{t+s}\boldsymbol{h}(\theta^{t+s}, z^{t+s})\right]$$

It is immediate that

$$\boldsymbol{w}(\theta^t, z^t) = \sum_{z^{t+1} \succ z^t} \pi_{t+1}(z^{t+1}|z^t) \sum_{\theta^{t+s} \succ \theta^t} f_{t+1}(\theta^{t+1}|\theta^t) \left[\boldsymbol{u}(\theta^{t+1}, z^{t+1}) - \theta_{t+1}\boldsymbol{h}(\theta^{t+1}, z^{t+1}) + \beta \boldsymbol{w}(\theta^{t+1}, z^{t+1})\right]$$

Let then

$$\hat{\boldsymbol{w}}(\theta^t, z^t|r) = \sum_{z^{t+1} \succ z^t} \pi_{t+1}(z^{t+1}|z^t) \sum_{\theta^{t+s} \succ \theta^t} f_{t+1}(\theta^{t+1}|\theta^t) \Big[\boldsymbol{u}(\theta^{t-1}, r, \theta_{t+1}, z^{t+1}) \\ - \theta_{t+1}\boldsymbol{h}(\theta^{t-1}, r, \theta_{t+1}, z^{t+1}) + \beta \boldsymbol{w}(\theta^{t-1}, r, \theta_{t+1}, z^{t+1})\Big]$$

denote the continuation utility of an agent who deviates only in period t, by announcing r, instead of her true period t shock, θ.

Incentive compatibility implies, for all $\theta^{t-1}, z^t, \theta, r \in \Theta$,

$$\boldsymbol{u}(\theta^t, z^t) - \theta \boldsymbol{h}(\theta^t, z^t) + \beta \boldsymbol{w}(\theta^t, z^t) \geq \boldsymbol{u}(\theta^{t-1}, r, z^t) - \theta \boldsymbol{h}(\theta^{t-1}, r, z^t) + \beta \boldsymbol{w}(\theta^{t-1}, r, z^t). \tag{5}$$

It is not hard to show – see Lemma 4, in the appendix — that (5) is also sufficient for incentive compatibility when idiosyncratic shocks follow a Markov process.

The last step toward our goal is to follow Phelan (1994) and use the multipliers for the resource constraints in the planner's Program $\mathcal{P}_0$ as a sequence of price functionals for consumption goods, $\{\mathcal{Q}_t(z^t)\}_t$. This defines an alternative program such that a unique budget inter-temporal budget constraint defined by

$$\sum_t \sum_{z^t} \mathcal{Q}_t(z^t) \left[\sum_{\theta^t} \mathcal{f}_t(\theta^t) \left[C(\boldsymbol{u}(\theta^t, z^t)) - N(\boldsymbol{h}(\theta^t, z^t)) z_t \right] \right] \geq 0$$

substitutes for the period by period, state by state resource constraints (2).

Let $\boldsymbol{w}^\star$ be the expected utility attained at the solution of the planner's program. Then, given $\{\mathcal{Q}_t(z^t)\}_t$ define program $\mathcal{P}_1$, as follows

$$\min_{(\boldsymbol{u},\boldsymbol{h})} \sum_t \sum_{z^t} \mathcal{Q}_t(z^t) \left[\sum_{\theta^t} \mathcal{f}_t(\theta^t) \left[C(\boldsymbol{u}(\theta^t, z^t)) - N(\boldsymbol{h}(\theta^t, z^t)) z_t \right] \right], \tag{6}$$

subject to

$$\boldsymbol{w}^\star = \sum_t \beta^{t-1} \sum_{z^t} \pi_t(z^t) \sum_{\theta^t} \mathcal{f}_t(\theta^t) \left[\boldsymbol{u}_t(\theta^t, z^t) - \theta_t \boldsymbol{h}_t(\theta^t, z^t) \right],$$

and (4).

Simple duality arguments allows us to show that (u^*, h^*) solves this cost minimization program for the prices defined in the previous paragraphs if and only if it solves the original program, $\mathcal{P}_0$. The cost minimization program, $\mathcal{P}_1$, now has an apparent recursive structure that the original program $\mathcal{P}_0$ lacks.

3.1.1. The i.i.d. case

We start our discussion with the case of i.i.d. idiosyncratic shocks. Let then $f(i)$ denote the probability that type $\theta_{(i)}$ is realized. The continuation utility, w, only depends on the history of reports made by the agent, not the actual history of her idiosyncratic shocks, under i.i.d. shocks. This greatly simplifies the exposition since a single state variable is needed: the promised continuation utility, w. In Section 3.1.2 we re-introduce persistence.

The cost minimization program (6) is separable in component planning programs defined in such a way that each component planner is responsible for delivering continuation utility $\boldsymbol{w}^\star$ at a minimum cost. Given $\boldsymbol{w}(\theta^t, z^t)$ (and $(\mathcal{Q}_t(z^t))_t$) we can find the optimal allocations following (θ^t, z^t) regardless of what is happening in the other nodes.

Although state prices $\mathcal{Q}_t(z^t)$ are a function of the history of aggregate shocks up to period t, these prices must take a very special form if allocations do not display memory. In Lemma 3 in the appendix we show that a necessary condition for the optimal allocation in problem (6) to be memoryless is that state prices have a multiplicative structure: $\mathcal{Q}_t(z^t|z^{t-1}) \equiv \mathcal{Q}_t(z^t)/\mathcal{Q}_{t-1}(z^{t-1}) = b_{t-1}(z_{t-1})a_t(z_t)$.

To give absence of memory a chance, we impose this structure on aggregate state prices in this section. Each component program has now a recursive structure which we explain next.

Let us start with period T. In this case, if the agent's promised utility is w, the component planner's program is simply

$$V_T(w) = \min_{(u_i(z), h_i(z))_{i,z}} \sum_z a_T(z) \sum_i f(i) \left[C(u_i(z)) - zN(h_i(z)) \right]$$

s.t.

$$u_i(z) - \theta_{(i)} h_i(z) \geq u_j(z) - \theta_{(i)} h_j(z)), \forall i, j, z$$

and

$$w = \sum_z \pi(z) \sum_i f(i) \left[u_i(z) - \theta_{(i)} h_i(z) \right]. \tag{7}$$

It is not hard to see that $V_T(\cdot)$ is a strictly increasing and strictly convex function.

For $t < T$, we write the component program in recursive form as follows,

$$V_t(w) = \min \sum_z a_t(z) \sum_i f(i) \left[C(u_i(z)) - zN(h_i(z)) + b_t(z) V_{t+1}(W_i(z)) \right] \tag{8}$$

subject to

$$u_i(z) - \theta_{(i)}h_i(z) + \beta W_i(z) \geq u_j(z) - \theta_{(i)}h_j(z) + \beta W_j(z), \forall i, j, z, \tag{9}$$

and

$$w = \sum_z \pi(z) \sum_i f(i)\left[u_i(z) - \theta_{(i)}h_i(z) + \beta W_i(z)\right]. \tag{10}$$

The policy functions $u_i(z)$, $h_i(z)$, and $W_i(z)$ are, respectively, the flow utility of consumption, the dis-utility of effort and the promised utility for an agent who announces to be of type $\theta_{(i)}$. These are functions of w and t as well, but we have omitted this dependence for economy. We have also dropped the time subscript from prices, $a(z)$ and $b(z)$.

The first order condition with respect to $W_i(z)$ is

$$a_t(z)f(i)b_t(z)V'_{t+1}(W_i(z)) + \beta \sum_j \{\alpha_{j,i} - \alpha_{i,j}\} = \pi(z)\lambda f(i)\beta, \tag{11}$$

where $\alpha_{i,j}$ is the Lagrange multiplier attached to the constraint that a type $\theta_{(j)}$ agent does not prefer to announce to be a $\theta_{(i)}$ type. Adding (11) across i, yields

$$b_t(z)a_t(z)\sum_i f(i)V'(W_i(z)) = \lambda\beta\pi(z),$$

If the allocation does not display memory, $W_i(z)$ must be independent of z for all i. Hence, if we define $\tilde{\lambda} := \lambda\pi(z)/(a_t(z)b_t(z))$, then (11) implies that $\tilde{\lambda}$ does not depend on z. But in this case, for $\tilde{\alpha}_{i,j} := \alpha_{i,j}/(b_t(z)a_t(z))$, we have that, for all i, $\sum_j\{\tilde{\alpha}_{i,j} - \tilde{\alpha}_{j,i}\}$ is independent of z.

With i.i.d. shocks only local downward constraints bind at the optimum. Then, $\sum_j\{\alpha_{i,j} - \alpha_{j,i}\} = \alpha_{i+1,i} - \alpha_{i,i-1}$, for all i. Moreover, for type θ_1, $\alpha_{i,i-1} = 0$, whereas for type θ_N, $\alpha_{i+1,i} = 0$. Hence, $\tilde{\alpha}_{i,j}$ is independent of z for all i, j. If (normalized) incentive and resource constraint multipliers are independent of the aggregate state, so are the optimal wedges.

Proposition 2. *Assume that idiosyncratic shocks are i.i.d., and that only local downward constraints bind, then the optimal allocation in problem (6) is memoryless only if labor wedges,*

$$\frac{\theta_{(i)}C'(u_i(z))}{zN'(h_i(z))} - 1$$

are state-invariant.

The proof of this proposition uses the first order conditions with respect to $u_i(z)$ and $h_i(z)$ and the fact that both $\tilde{\lambda}$ and $\tilde{\alpha}_{i,j}$ are independent of z when the allocation does not display memory to show that not only $C'(u_i(z))$ but also $N'(h_i(z))$ is separable. The result is immediate from there. We omit it for brevity.

Wedges are introduced to make false reports less attractive. Larger wedges, therefore, signal more redistribution. Proposition 2 therefore implies that, if the distribution of utility promises does not vary with the aggregate state, neither do wedges, and hence, current redistribution.

Proposition 3. *If preferences are of the form $\varphi(c) = \frac{[c-\bar{c}]^{1-\sigma}-1}{1-\sigma}$, for $\sigma \neq 1$, and $\zeta(n) = n^{\gamma}/\gamma$ then optimal allocations display memory of aggregate shocks.*

What is special about iso-elastic preferences is that the separability of $C'(u_i(z))$ and $N'(h_i(z))$ translates immediately into separability of $u_i(z)$ and $h_i(z)$, which is what we use to obtain the key characterization of the allocation used in our proof. There does not seem to be anything essential with iso-elastic preference for the result itself.

3.1.2. Persistent shocks

However convenient for analytical purposes, the assumption of i.i.d. shocks is probably not a good compromise if we want to think about the way productivity evolves along one's life. It is therefore important to allow idiosyncratic shocks to be persistent.

To simplify the exposition we follow Fernandes and Phelan (2000) and assume that there are only two productivity levels, and write the component program in recursive form as follows,

$$V_t(w, \hat{w}, k) = \min \sum_z a(z) \sum_i f(i|k)\Big[C(u_i(z)) - zN(h_i(z)) + b(z)V_{t+1}(W_i(z), \hat{W}_i(z), i)\Big], \tag{12}$$

subject to

$$u_i(z) - \theta_{(i)}h_i(z) + \beta W_i(z) \geq u_j(z) - \theta_{(i)}h_j(z) + \beta \hat{W}_j(z), \forall i, j, z. \tag{13}$$

$$w = \sum_z \pi(z) \sum_i f(i|k) \left[u_i(z) - \theta_{(i)} h_i(z) + \beta W_i(z) \right]. \tag{14}$$

and,

$$\hat{w} = \sum_z \pi(z) \sum_i f(i|l) \left[u_i(z) - \theta_{(i)} h_i(z) + \beta W_i(z) \right]. \tag{15}$$

When compared to the i.i.d. case, a type-l agent who deviates and announces to be k, obtains not only a temporary difference in utility with respect to the true k type, due to $\theta_{(l)} \neq \theta_{(k)}$, but also a difference in her continuation utility, due to $f(i|k) \neq f(i|l)$. The planner takes the latter into account trough the threat keeping constraint (15).

The same results found for i.i.d. shocks are valid here. Although the pattern of binding incentive compatibility constraints is harder to establish – see, e.g., Battaglini and Lamba (2015) –, the proofs for this case do not depend on it.[9]

3.2. The necessity of logarithmic utility

We have seen that any constrained efficient allocation must display memory of aggregate uncertainty, if preferences are iso-elastic and φ is not logarithmic. How about preferences which are not iso-elastic?

Unfortunately, a general answer is not possible without fully solving the program. Still, we can show that if preferences are such that φ is not logarithmic, then there is a configuration for the parameters of the economy for which efficiency requires memory.

To do this, we adopt the following definitions. Let Φ be the set of twice continuously differentiable functions $\varphi : \mathbb{R}_{++} \mapsto \mathbb{R}$, on $\mathbb{R}_+$ which satisfy $\varphi', -\varphi'' > 0$ and $\lim_{c \searrow 0} \varphi'(c) = \infty$. Its inverse is denoted C, which domain is the open interval I_φ.[10] Similarly, let ϑ be the set of twice continuously differentiable functions, $\zeta : \mathbb{R}_+ \mapsto \mathbb{R}$, satisfying $\zeta', \zeta'' > 0$ and $\zeta'(0) = \zeta(0) = 0$. Its inverse, N, has a domain $I_\zeta = [0, \infty)$. $\mathcal{Z}$ is the set of pairs (Z, π) where $Z \subset \mathbb{R}_{++}$is a finite set and $\pi \in \Delta\left(Z^T\right)$. Let also $\mathcal{F}$ be the set of triples $(T, \Theta, \mathbf{f})$, where $T \in \{2, 3, \ldots\}$, $\Theta := \left(\theta_{(1)}, \ldots, \theta_{(K)}\right) \subset \mathbb{R}_{++}$ is a finite set and $\mathbf{f} = \left(f_t\right)_{t=1}^T$ includes an initial distribution $f_1(\cdot) \in \Delta(\Theta)$ and transition probabilities $f_t(\cdot \mid \cdot)$ satisfying $f_t(\cdot \mid \theta) \in \Delta(\Theta)$, for any $\theta \in \Theta$ and $t \in \{2, .., T\}$. Finally, for each (φ, ζ), let U be the set of mappings $\boldsymbol{u} = \{\boldsymbol{u}_t\}_{t=1}^T$ with $\boldsymbol{u}_t : \Theta^t \times Z^t \mapsto I_\phi$, and H, the set of mappings $\boldsymbol{h} = \{\boldsymbol{h}_t\}_{t=1}^T$ with $\boldsymbol{h} : \Theta^t \times Z^t \mapsto I_\zeta$.

Define the program $\mathcal{P}(\varphi, \zeta, (Z, \pi), (T, \Theta, \mathbf{f}))$ as the maximization of (3), subject to (2), and (5). We may state the main result of this section.

Proposition 4. *If the solution to problem $\mathcal{P}(\varphi, \zeta, (Z, \pi), (T, \Theta, \mathbf{f}))$ is memoryless for any $(\zeta, (Z, \pi), (T, \Theta, \mathbf{f})) \in \vartheta \times \mathcal{Z} \times \mathcal{F}$, then, $\varphi \in \Phi$ satisfies $\varphi(c) = \ln c$.*

Hence, unless preferences for consumption are logarithmic, it is always possible to define an economy for which one can prove that the allocation displays memory.

4. Extensions

4.1. Aggregate impact on relative productivity

The multiplicative nature of aggregate shock has played an important role thus far, for it preserves the relative (across workers) cost of providing effort $\theta_{(i)}/\theta_{(j)}$. What happens with memory when this is not the case? Formally, what if z impacts the economy through individual effort costs $\theta_{(i)}(z)$ in such a way that there are two states, z and z', and two types such that $\theta_{(i)}(z)/\theta_{(j)}(z) \neq \theta_{(i)}(z')/\theta_{(j)}(z')$?

We assume that $\varphi(c) = \ln c$, $\zeta(n) = n^\gamma/\gamma$, and that there are only two idiosyncratic types: $\theta_{(1)}(z) = \theta_{(1)}$, $\theta_{(2)}(z) = z^{-1/\gamma}\theta_{(2)}$, $\theta_{(1)} < \theta_{(2)}$, $z < 1$.

Imposing the independence of continuation utility $W_i(z)$ on aggregate shock z allows us to work, as in Section 3.1.1, with the normalized Lagrange multipliers $\tilde{\lambda}$ and $\tilde{\alpha}$. These multipliers do not depend on z.

The first order conditions with respect to $u_i(z)$, $i = 1, 2$, imply that $u_i(z)$ is of the form $u_i(z) = \ln \delta_i + \ln z$.

As for $h_i(z)$, dividing the first order condition with respect to $h_1(z)$ by the first order condition with respect to $h_2(z)$, yields

$$\frac{h_1(z)}{h_2(z)} = \left[\frac{\tilde{\lambda}\theta_{(1)} - \tilde{\alpha}\theta_{(1)}/f_1}{\tilde{\lambda}\theta_{(1)} z^{1/\gamma} + \tilde{\alpha}\theta_{(1)}/f_2} \right]^{\frac{1}{\gamma-1}}. \tag{16}$$

[9] For the i.i.d. case, we use the fact that only local incentive constrains bind to show separability for the $N'(h)$ function. For the case with persistence a more direct proof is possible using the fact that each $\hat{W}_i(z)$ appears in only one incentive constraint.

[10] If $\inf I_\varphi > -\infty$, then with some abuse of notation, we assume that point $\inf I_\varphi$ is included in I_φ and satisfies $C\left(\inf I_\varphi\right) = 0$.

If the incentive constraint binds in two aggregate states,

$$u_1(z) - u_2(z) - \theta_{(1)}\left[h_1(z) - h_2(z)\right] = \beta\left[W_2 - W_1\right],$$

which in this case is

$$\ln\delta_1(w) - \ln\delta_2(w) - \theta_{(1)}\left[h_1(w) - h_2(w)\right] = \beta\left[W_2 - W_1\right].$$

This equality requires $h_1(w) - h_2(w)$ to be independent of z. Which from (16) cannot be the case.

What this example shows is that if the aggregate shock affects the relative productivity of agents, memory will be optimal due to the effect of aggregate shock on the cost of providing incentives. Werning (2007), who also considers this type of non-independence, shows that wedges vary with the aggregate state. Nonetheless, allocations remain independent of past aggregate shocks. This is mostly due to the fact that in Werning (2007) revelation of private takes place only in the first period. Of course the timing of the announcement in the first period matters for his independence results. If agents are asked their types before knowing the first realization of the aggregate state, then we have an allocation for which wedges vary across states, but there is no memory. Otherwise, the first period and only this period's aggregate shock, would affect future allocations. The same timing of aggregate shocks is found in Phelan (1994). Whereas independent shocks would lead to a memoryless allocation in his model, non-independence leads to a form of short lived memory in the form of Markovian state prices. The same timing is adopted by Scheuer (2013) who explores the role of interdependence of aggregate shocks and idiosyncratic shock distributions in the context of financial asset taxation.

4.2. Capital accumulation

Capital accumulation generates a form of history dependence which may interact with the one we have isolated here. A description of how this interaction plays out in full generality would lead us astray. Instead, we show how even when, absent capital accumulation, the constrained efficient allocations would not display memory, the presence of capital introduces such dependence.

We assume that shocks are i.i.d, $\varphi(c) = \ln c$, and the technology is such that aggregate output Y is produced by combining capital, K, and aggregate labor, $\mathcal{N}(z^t) = \sum_{\theta^t} f_t(\theta^t)N(\boldsymbol{h}(\theta^t, z^t))$, according to a Cobb–Douglas production function $Y_t = z_t K_t^a N_t^{1-a}$. We also assume full depreciation. Using $\mathcal{C}_t(z^t) = \sum_{\theta^t} f_t(\theta^t)C(\boldsymbol{u}(\theta^t, z^t))$ to denote aggregate consumption we write the economy's resource constraint as

$$\mathcal{C}_t(z^t) = z_t K(z^t)_t^a \mathcal{N}(z^t)_t^{1-a} - K_{t+1}(z^{t+1}).$$

Define the function $\Upsilon_t(\mathcal{C}_t(z^t))$ as the maximum expected utility that one can attain from consumption when aggregate consumption is $\mathcal{C}_t(z^t)$. Similarly define $\Gamma(\mathcal{N}_t(z^t))$ the minimum expected disutility of effort necessary to generate total labor supply $\mathcal{N}_t(z^t)$. Note that the distribution of consumption and effort is restricted by incentive compatibility.

It is not hard to check that $\Upsilon_t(\mathcal{C}_t(z^t))$ is of the form $\upsilon_t \ln\mathcal{C}_t(z^t)$, and Γ is convex in $\mathcal{N}_t(z^t)$. This suffices to show that K_{t+1} is a fixed fraction of Y_t, and for all t, $\mathcal{N}_t(z^t) = N_t$ for all z^t. Since the capital stock in t depends on z_{t-1}, the impact of a shock persists for one period.

5. Conclusion

Using a very simple model of the economy in the tradition of Lucas and Stokey (1983), we assess the consequences of allowing social insurance to vary optimally with the aggregate state. A common feature we find that optimal allocations possess is dependence on the history of aggregate shocks: temporary changes in aggregate productivity affects not only current wedges but also inequality in future periods. Trade-offs which arise from incentive provisions and risk sharing usually vary across states, hence the amount of incentives back loading, implying long run consequence of temporary aggregate shocks.

Our model shares some features already found in Werning (2007), where only redistribution is considered, with new ones. Specially, we consider whether current shocks should have any baring in long term inequality which we capture through the notion of memory of aggregate uncertainty.

The results herein raise important questions for future research regarding specific government policies, e.g., unemployment insurance, workfare, etc.

Appendix A

Proof of Proposition 3. We start with iso-elastic preferences, $\varphi(c) = c^{1-\rho}/(1-\rho)$, $\zeta(n) = n^\gamma/\gamma$, for which $C(u_i) = (\alpha u_i)^\alpha$, $N(h_i) = (\epsilon h_i)^\epsilon$. In this case by imposing lack of memory we have shown that

$$(\alpha u_i)^{\alpha-1} = \delta_i^u \varrho(z) \Rightarrow C(u_i) = \left[\delta_i^u \varrho(z)\right]^{\frac{\alpha}{\alpha-1}},$$

and

$$(\epsilon h_i)^{\epsilon-1} = \delta_i^h \frac{\varrho(z)}{z} \Rightarrow N(h_i) = \left[\delta_i^h \frac{\varrho(z)}{z}\right]^{\frac{\epsilon}{\epsilon-1}},$$

for some δ_i^u and δ_i^u which do not depend on z.

This is true for all w. That is,

$$\tilde{u}_i(w,z) = \left[\delta_i^u(w)\varrho(z)\right]^{\frac{1}{\alpha-1}}/\alpha \text{ and } \tilde{h}_i(w,z) = \left[\delta_i^h(w)\varrho(z)/z\right]^{\frac{1}{\epsilon-1}}/\epsilon.$$

With some abuse in notation we can write the resource constraint as

$$\sum_w \sum_i g(w) f(i) C(\tilde{u}_i(w,z)) = z \sum_w \sum_i g(w) f(i) N(\tilde{h}_i(w,z)),$$

where $g(w)$ is the share of agents who are entitled to continuation expected utility w. This can be used to solve for $\varrho(z)$: $\varrho(z) = z^{\frac{\alpha-1}{\alpha-\epsilon}}$, which implies

$$\frac{u_i}{h_i} = \frac{\alpha}{\epsilon} \frac{\left[\delta_i^u z^{\frac{\alpha-1}{\alpha-\epsilon}}\right]^{\frac{1}{\alpha-1}}}{\left[\delta_i^h z^{\frac{\epsilon-1}{\alpha-\epsilon}}\right]^{\frac{1}{\epsilon-1}}} = \frac{\alpha}{\epsilon} \frac{\left[\delta_i^u\right]^{\frac{1}{\alpha-1}}}{\left[\delta_i^h\right]^{\frac{1}{\epsilon-1}}}.$$

If the $j|i$ incentive constraint binds, then

$$u_i - \theta_{(i)} h_i + \beta W_i = u_j - \theta_{(i)} h_j + \beta W_j,$$

which in this case is $u_i - u_j - \theta_{(i)}\left[h_i - h_j\right] = \beta\left[W_j - W_i\right]$, or

$$u_i \left\{1 - \frac{u_j}{u_i} - \theta_{(i)}\left[\frac{h_i}{u_i} - \frac{u_j}{u_i}\frac{h_j}{u_j}\right]\right\} = \beta\left[W_j - W_i\right].$$

Using $\tilde{u}_i(w,z) = [\delta_i^u(x)\varrho(z)]^{\frac{1}{\alpha-1}}/\alpha$,

$$\left[\delta_i^u(w)\varrho(z)\right]^{\frac{1}{\alpha-1}} \left\{1 - \frac{u_j}{u_i} - \theta_{(i)}\left[\frac{h_i}{u_i} - \frac{u_j}{u_i}\frac{h_j}{u_j}\right]\right\} = \alpha\beta\left[W_j - W_i\right],$$

or

$$\left[\delta_i^u(w)\varrho(z)\right]^{\frac{1}{\alpha-1}} = \beta\left[W_j - W_i\right]\left[1 - \frac{u_j}{u_i} - \theta_{(i)}\left[\frac{h_i}{u_i} - \frac{u_j}{u_i}\frac{h_j}{u_j}\right]\right]^{-1}.$$

The left hand side depends on z, whereas the denominator on the right hand side does not. A contradiction with the assumption that W_i does not depend on z.

With utility from consumption of the form $\varphi(c) = (c-\bar{c})^{1-\rho}/(1-\rho)$, $u_i(w,z) = \left[\delta_i^u(w)\varrho(z)\right]^{\frac{1}{\alpha-1}}$, we have

$$\varrho(z)^{\frac{\alpha}{\alpha-1}} \sum_w \sum_i g(w) f(i) \left[\delta_i^u\right]^{\frac{\alpha}{\alpha-1}} - z\left[\frac{\varrho(z)}{z}\right]^{\frac{\epsilon}{\epsilon-1}} \sum_w \sum_i g(w) f(i) \left[\delta_i^h\right]^{\frac{\epsilon}{\epsilon-1}} = -\bar{c}.$$

Hence,

$$\varrho(z)^{\frac{\alpha}{\alpha-1}} A_u - \varrho(z)^{\frac{\epsilon}{\epsilon-1}} z^{-\frac{1}{\epsilon-1}} A_h = \bar{n}z - \bar{c}$$

To give this functional form a chance, let us assume that $\varrho(z)$ is well defined and differentiable. Then,

$$\frac{\alpha\varrho(z)^{\frac{\alpha}{\alpha-1}}}{\alpha-1}\frac{\varrho'(z)z}{\varrho(z)}A_u - \frac{\epsilon\varrho(z)^{\frac{\epsilon}{\epsilon-1}}}{\epsilon-1}\frac{\varrho'(z)z}{\varrho(z)}z^{-\frac{1}{\epsilon-1}}A_h + \frac{\varrho(z)^{\frac{\epsilon}{\epsilon-1}}}{\epsilon-1}z^{-\frac{1}{\epsilon-1}}A_h = 0. \tag{17}$$

Now, a binding incentive constraint implies

$$\varrho(z)^{\frac{1}{\alpha-1}} \varsigma_{ij}^u(w) - \varrho(z)^{\frac{1}{\epsilon-1}} z^{\frac{1}{1-\epsilon}} \varsigma_{ij}^h(w) = \varsigma_{ij}^W(w),$$

for

$$\varsigma_{ij}^u(w) = \left[\delta_i^u(w)\right]^{\frac{1}{\alpha-1}} - \left[\delta_j^u(w)\right]^{\frac{1}{\alpha-1}},$$

$$\varsigma_{ij}^h(w) = \theta_{(i)}\left\{\left[\delta_i^h(w)\right]^{\frac{1}{\epsilon-1}} - \left[\delta_j^h(w)\right]^{\frac{1}{\epsilon-1}}\right\},$$

and

$$\varsigma_{ij}^{W}(w) = \beta\left[W_j - W_i\right].$$

One can use this to find an expression,

$$\frac{\varrho'(z)z}{\varrho(z)} = \frac{\frac{1}{\epsilon-1}\varrho(z)^{\frac{1}{\epsilon-1}} z^{\frac{1}{1-\epsilon}} \varsigma_{ij}^{h}(w)}{\frac{1}{\alpha-1}\varrho(z)^{\frac{1}{\alpha-1}} \varsigma_{ij}^{u}(w) - \frac{1}{\epsilon-1}\varrho(z)^{\frac{1}{\epsilon-1}} z^{\frac{1}{1-\epsilon}} \varsigma_{ij}^{h}(w)},$$

for $\varrho'(z)z/\varrho(z)$ to substitute in (17).

Then,

$$\frac{\frac{1}{\epsilon-1}\varrho(z)^{\frac{\epsilon}{\epsilon-1}} z^{-\frac{1}{\epsilon-1}} A_h}{\frac{\alpha}{\alpha-1}\varrho(z)^{\frac{\alpha}{\alpha-1}} A_u - \frac{\epsilon}{\epsilon-1}\varrho(z)^{\frac{\epsilon}{\epsilon-1}} z^{-\frac{1}{\epsilon-1}} A_h} = \frac{\frac{1}{\epsilon-1}\varrho(z)^{\frac{1}{\epsilon-1}} z^{\frac{1}{1-\epsilon}} \varsigma_{ij}^{h}(w)}{\frac{1}{\alpha-1}\varrho(z)^{\frac{1}{\alpha-1}} \varsigma_{ij}^{u}(w) - \frac{1}{\epsilon-1}\varrho(z)^{\frac{1}{\epsilon-1}} z^{\frac{1}{1-\epsilon}} \varsigma_{ij}^{h}(w)},$$

which finally yields

$$\frac{\varrho(z)^{\frac{\epsilon}{\epsilon-1}} z^{-\frac{1}{\epsilon-1}}}{\varrho(z)^{\frac{\alpha}{\alpha-1}}} = (\alpha-1)^{-1}\left[\alpha\frac{A_u}{A_h} - \frac{\varsigma_{ij}^{u}(w)}{\varsigma_{ij}^{h}(w)}\right].$$

The right hand side is independent of z. So must the left hand side be. Solving for $\varrho(z)$ allows one to see that it is the same as if $\bar{c} = 0$ □

Proof of Proposition 4. We prove the contrapositive statement. First, we assume that $\varphi(\cdot) \neq \ln(\cdot)$ and, using Lemma 5, that there exists $\boldsymbol{u}_0, \varepsilon, a \in \mathbb{R}_{++}$ such that, for any $\boldsymbol{u} \in B_\varepsilon(\boldsymbol{u}_0)$,

$$\frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})} - \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} > a \text{ and } \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} \text{ is strictly increasing.}$$

The other case considered in Lemma 5 is easily handled with the same argument.

Consider $\zeta_0(n) = n^2/2$ and $T = 2$. Find $Z = \{z_0\}$ and $\Theta = \{\theta_0\}$ such that $\mathcal{P}(\varphi, \zeta, (\{z_0\}, \pi_0), (T, \{\theta_0\}, \mathbf{f}_0))$ has a solution with $\boldsymbol{h}_t = \boldsymbol{h}_0 \in \mathbb{R}_{++}$ and $\boldsymbol{u}_t = \boldsymbol{u}_0$ satisfying $C(\boldsymbol{u}_0) = c_0$. This is, for example, the case if $z_0 = \frac{C(\boldsymbol{u}_0)}{\sqrt{2}}$ and $\theta_0 = \frac{z_0}{C'(\boldsymbol{u}_0)\sqrt{2}}$.

Now, for any $(z, \theta') \in \mathbb{R}_{++} \times (0, \theta_0)$, a necessary and sufficient condition for an interior solution for the problem $\mathcal{P}(\varphi, \zeta, (\{z\}, \pi^z), (T, \{\theta_0, \theta'\}, \mathbf{f}))$, where $\mathbf{f}$ satisfies $f_1(\theta_0) = f_1(\theta') = \frac{1}{2}$ and $f_2(\theta_0 \mid \theta_1) = 1$ and π^z assigns probability one in z for period 1 and probability one in z_0 for period 2, is the existence of $(\lambda_1, \lambda_2, \mu) \in \mathbb{R}_+ \setminus \{(0,0,0)\}$ such that

$$\frac{1}{2} - \lambda_1\frac{1}{2}C'(\boldsymbol{u}_1(\theta_0)) - \mu_1 = 0, \tag{18}$$

$$\frac{1}{2} - \lambda_1\frac{1}{2}C'(\boldsymbol{u}_1(\theta')) + \mu_1 = 0, \tag{19}$$

$$-\theta_0\frac{1}{2} + z\lambda_1\frac{1}{2}N'(\boldsymbol{h}_1(\theta_0)) + \mu_1\theta' = 0, \tag{20}$$

$$-\theta'\frac{1}{2} + z\lambda_1\frac{1}{2}N'(\boldsymbol{h}_1(\theta')) - \mu_1\theta' = 0, \tag{21}$$

$$\beta\frac{1}{2} - \lambda_2\frac{1}{2}C'(\boldsymbol{u}_2(\theta_0, \theta_0)) - \mu_1\beta = 0, \tag{22}$$

$$\beta\frac{1}{2} - \lambda_2\frac{1}{2}C'(\boldsymbol{u}_2(\theta', \theta_0)) + \mu_1\beta = 0, \tag{23}$$

$$-\beta\theta_0\frac{1}{2} + z_0\lambda_2\frac{1}{2}N'(\boldsymbol{h}_2(\theta_0, \theta_0)) + \mu_1\beta\theta_0 = 0, \tag{24}$$

$$-\theta_0\frac{1}{2} + z_0\lambda_2\frac{1}{2}N'(\boldsymbol{h}_2(\theta', \theta_0)) - \mu_1\beta\theta_0 = 0, \tag{25}$$

$$C(\boldsymbol{u}_1(\theta_0)) + C(\boldsymbol{u}_1(\theta')) = z\left[N(\boldsymbol{h}_1(\theta_0)) + N(\boldsymbol{h}_1(\theta'))\right], \tag{26}$$

$$C(\boldsymbol{u}_2(\theta_0, \theta_0)) + C(\boldsymbol{u}_2(\theta', \theta_0)) = z_0\left[N(\boldsymbol{h}_2(\theta_0, \theta_0)) + N(\boldsymbol{h}_2(\theta', \theta_0))\right], \tag{27}$$

and

$$\boldsymbol{u}_1\left(\theta'\right)-\theta'\boldsymbol{h}_1\left(\theta'\right)+\beta\left[\boldsymbol{u}_2\left(\theta',\theta_0\right)-\theta_0\boldsymbol{h}_2\left(\theta',\theta_0\right)\right]$$
$$=\boldsymbol{u}_1\left(\theta_0\right)-\theta'\boldsymbol{h}_1\left(\theta_0\right)+\beta\left[\boldsymbol{u}_2\left(\theta_0,\theta_0\right)-\theta_0\boldsymbol{h}_2\left(\theta_0,\theta_0\right)\right]. \quad (28)$$

If $\theta'=\theta_0$ and $z=z_0$, this system has a solution with $\boldsymbol{u}_1(\cdot)=\boldsymbol{u}_2(\cdot)=\boldsymbol{u}_0$, $\boldsymbol{h}_1(\cdot)=\boldsymbol{h}_1(\cdot)=\boldsymbol{h}_0$, $\mu_1=0$ and $\lambda_1=\lambda_2=\left[C'(\boldsymbol{u}_0)\right]^{-1}$.

The implicit function theorem implies that there exists $\varepsilon_1>0$ such that this system has a solution

$$\left(\boldsymbol{u}_1^{(z,\theta')}(\cdot),\boldsymbol{u}_2^{(z,\theta')}(\cdot),\boldsymbol{h}_1^{(z,\theta')}(\cdot),\boldsymbol{h}_2^{(z,\theta')}(\cdot),\lambda_1^{(z,\theta')},\lambda_2^{(z,\theta')},\mu_1^{(z,\theta')}\right)$$

if $\max\left\{|z-z_0|,\left|\theta'-\theta_0\right|\right\}<\varepsilon_1$ and this solution is continuously differentiable in $\left(z,\theta'\right)$.[11]

We now show that, if $\max\left\{|z-z_0|,\left|\theta'-\theta_0\right|\right\}<\varepsilon_1$, then $(\boldsymbol{u}_2^{(z,\theta')}(\cdot),\boldsymbol{u}_2^{(z,\theta')}(\cdot))$ depends on z.

If were not the case, then, for any $\left(z,\theta'\right)$ satisfying $\max\left\{|z-z_0|,\left|\theta'-\theta_0\right|\right\}<\varepsilon_1$, we would have

$$\left(\boldsymbol{u}_2^{(z,\theta')}(\cdot),\boldsymbol{h}_2^{(z,\theta')}(\cdot)\right)=\left(\boldsymbol{u}_2^{(z_0,\theta')}(\cdot),\boldsymbol{h}_2^{(z_0,\theta')}(\cdot)\right).$$

Using the equations above we have that, for any $\left(z,\theta'\right)$ such that $\theta'\in(0,\theta_0)$ and $\max\left\{|z-z_0|,\left|\theta'-\theta_0\right|\right\}<\varepsilon_1$,

$$\beta\frac{1}{2}-\lambda_2^{(z,\theta')}\frac{1}{2}C'\left(\boldsymbol{u}_2^{(z,\theta')}(\theta_0,\theta_0)\right)-\mu_1^{(z,\theta')}\beta=0,$$

and

$$\beta\frac{1}{2}-\lambda_2^{(z,\theta')}\frac{1}{2}C'\left(\boldsymbol{u}_2^{(z,\theta')}\left(\theta',\theta_0\right)\right)+\mu_1^{(z,\theta')}\beta=0.$$

This implies

$$\lambda_2^{(z,\theta')}=\beta\left[C'\left(\boldsymbol{u}_2^{(z,\theta')}(\theta_0,\theta_0)\right)+C'\left(\boldsymbol{u}_2^{(z,\theta')}\left(\theta',\theta_0\right)\right)\right]^{-1},$$

and

$$\mu_1^{(z,\theta')}=\frac{1}{2}-\lambda_2^{(z,\theta')}\frac{1}{2\beta}C'\left(\boldsymbol{u}_2^{(z,\theta')}(\theta_0,\theta_0)\right).$$

Hence, if $\boldsymbol{u}_2^{(z,\theta')}=\boldsymbol{u}_2^{(z_0,\theta')}$, then $\mu_1^{(z,\theta')}=\mu_1^{(z_0,\theta')}$ and $\lambda_2^{(z,\theta')}=\lambda_2^{(z_0,\theta')}$.

Defining $d\lambda_1^{(z,\theta')}=\partial\lambda_1^{(z,\theta')}/\partial z$, and $\left(d\boldsymbol{u}_1^{(z,\theta')}(\cdot),d\boldsymbol{u}_2^{(z,\theta')}(\cdot),d\boldsymbol{h}_1^{(z,\theta')}(\cdot),d\boldsymbol{h}_2^{(z,\theta')}(\cdot)\right)$ analogously, we have

$$C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)d\boldsymbol{u}_1^{(z,\theta')}(\theta_0)+C'\left(\boldsymbol{u}_1^{(z,\theta')}\left(\theta'\right)\right)d\boldsymbol{u}_1^{(z,\theta')}\left(\theta'\right)$$
$$=N\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)+N\left(\boldsymbol{h}_1^{(z,\theta')}\left(\theta'\right)\right)+zN'\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)d\boldsymbol{h}_1^{(z,\theta')}(\theta_0)+zN'\left(\boldsymbol{u}h_1^{(z,\theta')}\left(\theta'\right)\right)d\boldsymbol{h}_1^{(z,\theta')}\left(\theta'\right),$$

which, using (18)–(21), yields

$$-\frac{d\lambda_1^{(z,\theta')}}{\lambda_1^{(z,\theta')}}=\frac{N\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)+N\left(\boldsymbol{h}_1^{(z,\theta')}\left(\theta'\right)\right)-\left[\dfrac{N'\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)^2}{N''\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)}+\dfrac{N'\left(\boldsymbol{h}_1^{(z,\theta')}\left(\theta'\right)\right)^2}{N''\left(h_1^{(z,\theta')}\left(\theta'\right)\right)}\right]}{\dfrac{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)^2}{C''\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)}+\dfrac{C'\left(\boldsymbol{u}_1^{(z,\theta')}\left(\theta'\right)\right)^2}{C''\left(\boldsymbol{u}_1^{(z,\theta')}\left(\theta'\right)\right)}-z\left[\dfrac{N'\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)^2}{N''\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)}+\dfrac{N'\left(\boldsymbol{h}_1^{(z,\theta')}\left(\theta'\right)\right)^2}{N''\left(\boldsymbol{h}_1^{(z,\theta')}\left(\theta'\right)\right)}\right]}.$$

[11] It's easy to show this system of equations has full rank when $\theta'=\theta_0$ and $z=z_0$.

This equation implies that $d\lambda_1^{(z,\theta')} < 0$, and

$$z d\lambda_1^{(z,\theta')} + \lambda_1^{(z,\theta')} = \left\{ C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)\left[\frac{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)}{C''\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)} - \frac{C\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)}{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)}\right] + C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)\left[\frac{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)}{C''\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)} - \frac{C\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)}{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)}\right]\right\} \left\{\frac{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)^2}{C''\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)} + \frac{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)^2}{C''\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)} - z\left[\frac{N'\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)^2}{N''\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)} + \frac{N'\left(\boldsymbol{h}_1^{(z,\theta')}(\theta')\right)^2}{N''\left(\boldsymbol{h}_1^{(z,\theta')}(\theta')\right)}\right]\right\}^{-1}. \tag{29}$$

We know that the incentive compatibility constraint of an agent of type θ' misreporting her type always binds if $\theta' < \theta_0$. This implies $\boldsymbol{u}_1^{(z,\theta')}(\theta') > \boldsymbol{u}_1^{(z,\theta')}(\theta_0)$ and $\boldsymbol{h}_1^{(z,\theta')}(\theta') > \boldsymbol{h}_1^{(z,\theta')}(\theta_0)$.

Using the fact that $(\boldsymbol{u}_2^{(z,\theta')}(\cdot), \boldsymbol{h}_2^{(z,\theta')}(\cdot)) = (\boldsymbol{u}_2^{(z_0,\theta')}(\cdot), \boldsymbol{h}_2^{(z_0,\theta')}(\cdot))$ for any (z,θ') such that $\theta' \in (0,\theta_0)$ and $\max\{|z-z_0|, |\theta'-\theta_0|\} < \varepsilon_1$, it must be that

$$\frac{d}{dz}\left\{\boldsymbol{u}_1^{(z,\theta')}(\theta') - \boldsymbol{u}_1^{(z,\theta')}(\theta_0) - \theta'\left[\boldsymbol{h}_1^{(z,\theta')}(\theta') - \boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right]\right\} = 0.$$

But, using the results obtained above,

$$\frac{d}{dz}\left[\boldsymbol{u}_1^{(z,\theta')}(\theta') - \boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right] = -\frac{d\lambda_1}{\lambda_1}\left[\frac{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)}{C''\left(\boldsymbol{u}_1^{(z,\theta')}(\theta')\right)} - \frac{C'\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)}{C''\left(\boldsymbol{u}_1^{(z,\theta')}(\theta_0)\right)}\right] > 0.$$

Note also that

$$\begin{aligned} &-\theta'\frac{d}{dz}\left[\boldsymbol{h}_1^{(z,\theta')}(\theta') - \boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right] \\ &= \theta'\frac{\left(z d\lambda_1^{(z,\theta')} + \lambda_1^{(z,\theta')}\right)}{z\lambda_1}\left[\frac{N'\left(\boldsymbol{h}_1^{(z,\theta')}(\theta')\right)}{N''\left(\boldsymbol{h}_1^{(z,\theta')}(\theta')\right)} - \frac{N'\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)}{N''\left(\boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right)}\right] \\ &= -2\theta'\frac{\left(z d\lambda_1^{(z,\theta')} + \lambda_1^{(z,\theta')}\right)}{z\lambda_1}\left[\boldsymbol{h}_1^{(z,\theta')}(\theta') - \boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right]. \end{aligned}$$

Using (29) and the assumption that

$$\frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})} - \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} > 0,$$

we have $z d\lambda_1^{(z,\theta')} + \lambda_1^{(z,\theta')} < 0$. Hence, we conclude that

$$\frac{d}{dz}\left\{\boldsymbol{u}_1^{(z,\theta')}(\theta') - \boldsymbol{u}_1^{(z,\theta')}(\theta_0) - \theta'\left[\boldsymbol{h}_1^{(z,\theta')}(\theta') - \boldsymbol{h}_1^{(z,\theta')}(\theta_0)\right]\right\} > 0.$$

A contradiction.

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There exists, in this case, at least one pair $(z_1, z_2) \in B_{\varepsilon_1}(z_0)$ and a set of productivity types $0 < \theta' < \theta_0$ such that

$$\left(\boldsymbol{u}_2^{(z_1,\theta')}(\cdot), \boldsymbol{h}_2^{(z_1,\theta')}(\cdot)\right) \neq \left(\boldsymbol{u}_2^{(z_2,\theta')}(\cdot), \boldsymbol{h}_2^{(z_2,\theta')}(\cdot)\right).$$

Finally, consider the problem $\mathcal{P}\left(\varphi, \zeta, \left(\{z_1, z_2\}, \pi'\right), \left(T, \{\theta_0, \theta'\}, \mathbf{f}\right)\right)$, where $\mathbf{f}$ satisfies $f_1(\theta_0) = f_1(\theta') = \frac{1}{2}$ and $f_2(\theta_0 \mid \theta_1) = 1$ and π' places equal probabilities on public shocks z_1 and z_2 in period 1, and probability one on public shock z_0 in period two. The solution of this problem, referred to as $(\boldsymbol{u}^*, \boldsymbol{h}^*)$ necessarily satisfies

$$\left(\boldsymbol{u}_2^*\left(\theta^2, (z, z_0)\right), \boldsymbol{h}_2^*\left(\theta^2, (z, z_0)\right)\right) = \left(\boldsymbol{u}_2^{(z,\theta')}(\theta_1), \boldsymbol{h}_2^{(z,\theta')}(\theta_1)\right),$$

for $z \in \{z_1, z_2\}$ and $\theta^2 = (\theta_1, \theta_0)$ such that $\theta_1 \in \{\theta_0, \theta'\}$. Hence, the solution for this problem displays memory of aggregate shocks. □

A.1. Lemmata

Lemma 2. *At the optimum,*

$$\mathbb{E}\left[\frac{C'(\boldsymbol{u}_{t+s}(\theta^{t+s}, z^{t+s}))}{C'(\boldsymbol{u}_t(\theta^t, z^t))}\middle|\theta^t\right] = \mathbb{E}\left[\frac{C'(\boldsymbol{u}_{t+s}(\hat{\theta}^{t+s}, z^{t+s}))}{C'(\boldsymbol{u}_t(\hat{\theta}^t, z^t))}\middle|\hat{\theta}^t\right],$$

for every, θ^t, $\hat{\theta}^t$, z^t and $z^{t+s} \succ z^t$.

Proof of Lemma 2. Let $\{\boldsymbol{u}, \boldsymbol{h}\}$ be our candidate optimal allocation. We shall construct a new allocation $\{\hat{\boldsymbol{u}}, \hat{\boldsymbol{h}}\}$ as follows. Increase consumption in period t, public history z^t, for agents with history θ^t in such a way that $\boldsymbol{u}(\theta^t, z^t) = \boldsymbol{u}(\theta^t, z^t) + \delta$. Because aggregate resources are fixed, this can only be done by reducing consumption for other individuals, say, those with history $\bar{\theta}^t$. The effect of such change in those agents' utilities is to reduce it by $\varepsilon(\delta)$. That is, $\varphi(\hat{\mathbf{c}}(\bar{\theta}^t, z^t)) = \varphi(\mathbf{c}(\bar{\theta}^t, z^t)) - \varepsilon(\delta)$. For small enough δ, the two are related by $\varepsilon(0) = 0$ and

$$\varepsilon(\delta) \approx \frac{\varphi'(\mathbf{c}(\bar{\theta}^t, z^t))}{\varphi'(\mathbf{c}(\theta^t, z^t))} \frac{f(\theta)}{f(\bar{\theta})} \delta. \tag{30}$$

For this reform not to have impact on incentives or on any individual's utility, it must compensate the change in utility for every continuation history.

Hence, fix some $z^{t+s} \succ z^t$, and for every $\theta^{t+s} \succ \theta^t$, let

$$\varphi(\hat{\mathbf{c}}(\theta^{t+s}, z^{t+s})) = \varphi(\mathbf{c}(\theta^{t+s}, z^{t+s})) - \frac{\delta}{\pi(z^{t+s}|z^t)\beta^s}.$$

Similarly, for every $\bar{\theta}^{t+s} \succ \bar{\theta}^t$, let

$$\varphi(\hat{\mathbf{c}}(\bar{\theta}^{t+s}, z^{t+s})) = \varphi(\mathbf{c}(\bar{\theta}^{t+s}, z^{t+s})) + \frac{\varepsilon(\delta)}{\pi(z^{t+s}|z^t)\beta^s}.$$

For every z^{t+s}, the marginal cost of such reform is

$$-\delta \sum_{\theta^{t+s} \succ \theta^t} \frac{f_{t+s}(\theta^{t+s-1}, \theta|\theta^t)}{\varphi'(\mathbf{c}(\theta^{t+s-1}, \theta, z^{t+s}))} + \varepsilon(\delta) \sum_{\tilde{\theta}^{t+s} \succ \tilde{\theta}^t} \frac{f_{t+s}(\tilde{\theta}^{t+s-1}, \theta|\tilde{\theta}^t)}{\varphi'(\mathbf{c}(\tilde{\theta}^{t+s-1}, \theta, z^{t+s}))}. \tag{31}$$

Substituting (30) in (31) it is clear that, unless

$$\sum_{\theta^{t+s} \succ \theta^t} \frac{\varphi'(\mathbf{c}(\theta^{t+s-1}, \theta, z^{t+s}))}{\varphi'(\mathbf{c}(\theta^{t+s-1}, \theta, z^{t+s}))} f_{t+s}(\theta^{t+s-1}, \theta|\theta^t) = \sum_{\tilde{\theta}^{t+s} \succ \tilde{\theta}^t} \frac{\varphi'(\mathbf{c}(\tilde{\theta}^{t+s-1}, \theta, z^{t+s}))}{\varphi'(\mathbf{c}(\tilde{\theta}^{t+s-1}, \theta, z^{t+s}))} f_{t+s}(\tilde{\theta}^{t+s-1}, \theta|\tilde{\theta}^t),$$

for all $\theta^t, \bar{\theta}^t, z^t, z^{t+s}, t$, and s, there is a reform that preserves utility and incentive compatibility and saves resources. □

Proof of Lemma 1. From Lemma 2 we have

$$\frac{C'(\boldsymbol{u}_t(\hat{\theta}^t, z^t))}{C'(\boldsymbol{u}_t(\theta^t, z^t))} = \frac{\mathbb{E}\left[C'(\boldsymbol{u}_{t+s}(\hat{\theta}^{t+s}, z^{t+s}))\middle|\hat{\theta}^t\right]}{\mathbb{E}\left[C'(\boldsymbol{u}_{t+s}(\theta^{t+s}, z^{t+s}))\middle|\theta^t\right]}$$

for every, θ^t, $\hat{\theta}^t$, $z^t$and $z^{t+s} \succ z^t$. By assumption, the right hand side does not depend on z^t. Thus, the right hand side cannot depend on z either. Then,

$$\frac{C'(\boldsymbol{u}_t(\hat{\theta}^t, z^t))}{C'(\boldsymbol{u}_t(\theta^t, z^t))} = \frac{\varphi_t^u(\theta^t)}{\varphi_t^u(\hat{\theta}^t)}.$$

History independence finally implies that $C'(\boldsymbol{u}_t(\hat{\theta}^t, z^t)) = \varphi_t^u(\theta^t)\eta_t(z_t)$. □

Lemma 3. *If the allocations does not display memory then, for every z^t, z^{t+1} $t \le T-1$, there are functions $a_{t+1}(z_{t+1})$ and $b_t(z_t)$ such that $\mathcal{Q}(z^{t+1}|z^t) = b_t(z_t)a_{t+1}(z_{t+1})$.*

Proof. Let $(\boldsymbol{u}_t^\star(\theta^t, z^t), \boldsymbol{h}_t^\star(\theta^t, z^t))$ solve the cost minimization program $\mathcal{P}_1$. We shall construct a new allocation $\{\boldsymbol{u}, \boldsymbol{h}\}$ as follows. For all t, θ^t, z^t, $\boldsymbol{h}_t(\theta^t, z^t) = \boldsymbol{h}_t^\star(\theta^t, z^t)$. For $\boldsymbol{u}$, increase flow utility from consumption in period t, public history z^t, for agents with history θ^t in such a way that $\boldsymbol{u}(\theta^t, z^t) = \boldsymbol{u}^\star(\theta^t, z^t) + \delta$. Fix an aggregate continuation $z^{t+1} \succ z^t$. Then for every continuation $\theta^{t+1} \succ \theta^t$ make $\boldsymbol{u}(\theta^{t+1}, z^{t+1}) = \boldsymbol{u}^\star(\theta^{t+1}, z^{t+1}) - \delta/(\beta\pi(z^{t+1}|z^t))$. This reform does not change w nor does it have any impact on incentive compatibility. The cost of such reform is, for 'small' δ

$$C'(\boldsymbol{u}^\star(\theta^t, z^t))\delta - \frac{\mathcal{Q}(z^{t+1}|z^t)}{\beta\pi(z^{t+1}|z^t)} \sum_{\theta^{t+1}\succ\theta^t} f_{t+1}(\theta^{t+1}|\theta^t)C'(\boldsymbol{u}^\star(\theta^{t+1}, z^{t+1}))\delta$$

Cost minimization then implies that there is no δ for which this is not zero, i.e.,

$$\mathcal{Q}(z^{t+1}|z^t) = \beta\pi(z^{t+1}|z^t)\left[\sum_{\theta^{t+1}\succ\theta^t} f_{t+1}(\theta^{t+1}|\theta^t)\frac{C'(\boldsymbol{u}^\star(\theta^{t+1}, z^{t+1}))}{C'(\boldsymbol{u}^\star(\theta^t, z^t))}\right]^{-1} \quad \square$$

If the allocations do not display memory, we know by Lemma 2 that $C'(\boldsymbol{u}_t(\hat{\theta}^t, z^t)) = \varphi_t^u(\theta^t)\eta_t(z_t)$. Hence,

$$\mathcal{Q}(z^{t+1}|z^t) = \beta\eta_t(z_t)\varphi_t^u(\theta^t)\frac{\pi(z_{t+1})}{\eta_{t+1}(z_{t+1})}\left[\sum_{\theta^{t+1}\succ\theta^t} f_{t+1}(\theta^{t+1}|\theta^t)\varphi_{t+1}^u(\theta^{t+1})\right]^{-1}.$$

Lemma 4. *An allocation is incentive compatible if and only if it satisfies (5).*

Proof. To show that (4) implies (5), assume that there is a one period-one state deviation that yields more expected utility following a history that occurs with positive probability. Then, the strategy of telling the truth in all nodes except the one in which one period deviation is welfare increasing; i.e., yields a larger expected utility than truth-telling. This means that constraint (4) is violated.

To prove the converse, assume that (4) is violated, i.e., there is a strategy σ that yields more expected utility than the truthful strategy. What we show is that it is possible to define a strategy that only violates one constraint of the type (5).

Toward this end, define τ, z^τ as the last node for which strategy σ recommends lying. Let $\sigma_{\tau-1}(\theta^{\tau-1}) = \tilde{\theta}^{\tau-1}$ and note that we can assume without loss that

$$\begin{aligned} &\boldsymbol{u}(\sigma(\theta^\tau, z^\tau), z^\tau) - \theta_\tau\boldsymbol{h}(\sigma(\theta^\tau, z^\tau), z^\tau) + \beta\boldsymbol{w}^c(\theta^\tau, z^\tau) \\ &\quad > \boldsymbol{u}((\tilde{\theta}^{\tau-1}, \theta_\tau), z^\tau) - \theta_\tau\boldsymbol{h}((\tilde{\theta}^{\tau-1}, \theta_\tau), z^\tau) + \beta\boldsymbol{w}((\tilde{\theta}^{\tau-1}, \theta_\tau), z^\tau) \end{aligned} \tag{32}$$

if $\tau < T$ or

$$\boldsymbol{u}(\sigma(\theta^\tau, z^\tau), z^\tau) - \theta_\tau\boldsymbol{h}(\sigma(\theta^\tau, z^\tau), z^\tau) > \boldsymbol{u}((\tilde{\theta}^{\tau-1}, \theta_\tau), z^\tau) - \theta_\tau\boldsymbol{h}((\tilde{\theta}^{\tau-1}, \theta_\tau), z^\tau)$$

if $\tau = T$.

To understand why this is without loss just note that if this condition is violated, then the strategy of following σ until the $z^{\tau-1} \prec z^\tau$ and telling the truth from τ, z^τ on is weakly better than following σ. Therefore, we may as well focus on this latter strategy.

Now consider the strategy of announcing the truth unless state $((\tilde{\theta}^{\tau-1}, \theta_\tau), z^\tau)$ is realized, in which case, one announces $\sigma((\tilde{\theta}^{\tau-1}, \theta_\tau), z^\tau)$. This is a one state deviation strategy that improves on truth-telling. It thus violate (5). □

Lemma 5. *Assume $\varphi(c) \neq \ln c$. Then, there exists $\bar{\boldsymbol{u}}$, ε, $a \in \mathbb{R}_{++}$ such that, for any $\boldsymbol{u} \in B_\varepsilon(\bar{\boldsymbol{u}})$,*[12]

$$\frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})} - \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} > a \text{ and } \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} \text{ is strictly increasing,}$$

[12] Of course the same is true if $\varphi(c) \neq k_1 \ln c + k_2$, for some $k_1 > 0$ and $k_2 \in \mathbb{R}$, $\varphi(c) \neq k_1 \ln c + k_2$.

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or

$$\frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})} - \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} < -a \text{ and } \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} \text{ is strictly decreasing.}$$

Proof. Notice that

$$\frac{d}{d\boldsymbol{u}}\left[\frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})}\right] = \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})}\left[\frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})} - \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})}\right], \tag{33}$$

so that

$$\frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})} - \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} > 0$$

implies that the function $\frac{C'(\cdot)}{C(\cdot)}$ is strictly increasing on an open neighborhood of $\boldsymbol{u}$.

Hence, if

$$\frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})} - \frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} = 0$$

for $\boldsymbol{u} \in [a, b]$ with $0 < a < b$, then (33) implies that $C'(\boldsymbol{u})/C(\boldsymbol{u}) = K$, for some constant $K > 0$. This means that, for any $\boldsymbol{u} \in [a, b]$, $C(\boldsymbol{u}) = \left[C(a)\, e^{-Ka}\right] e^{K\boldsymbol{u}}$.

We then conclude that, if the utility function $\varphi(\cdot)$ is not logarithmic (and hence its inverse is not exponential), then there exists at least one point $\bar{\boldsymbol{u}} \in \mathbb{R}_+$ such that

$$\frac{C''(\bar{\boldsymbol{u}})}{C'(\bar{\boldsymbol{u}})} - \frac{C'(\bar{\boldsymbol{u}})}{C(\bar{\boldsymbol{u}})} \neq 0.$$

Suppose that there exists $\bar{\boldsymbol{u}}, \varepsilon, a \in \mathbb{R}_{++}$ such that

$$\frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} - \frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})} > a$$

for any $\boldsymbol{u} \in B_\varepsilon(\bar{\boldsymbol{u}})$. Equation (33) implies that $\frac{C'(\cdot)}{C(\cdot)}$ is strictly increasing in $B_\varepsilon(\bar{\boldsymbol{u}})$. The opposite holds if

$$\frac{C'(\boldsymbol{u})}{C(\boldsymbol{u})} - \frac{C''(\boldsymbol{u})}{C'(\boldsymbol{u})}$$

is negative. □

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