Cost heterogeneity and Competitive Insurance*

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1 Introduction

In this note we study an extension of the competitive insurance model presented in Farinha Luz (2016) (henceforth FL) with the inclusion of heterogeneous fixed costs of insurance provision. For sufficiently small cost heterogeneity, we characterize Nash equilibria that approximate the mixed equilibria discussed in FL.

This note is organized as follows. Section 2 presents the extension of FL's model. Section 3 presents necessary conditions for an ordered equilibrium that define a differential system. The connection between this differential system and the mixed equilibrium presented in FL is discussed in Section 4. In Section 5 we show that the differential equations described are also sufficient for an equilibrium. Finally, Section 6 highlights the key properties of pure strategy equilibria in this model while Section 7 concludes. Some proofs are relegated to the appendix.

2 Extended model

We consider the model described in FL with the presence of a fixed cost to serve an agent of any risk type and focusing on the case of duopoly.[1] We assume that the fixed cost of each firm, denoted as ψ, is independently drawn from a uniform distribution on the interval $\left[0, \bar{\psi}\right]$.[2] This cost distribution is equivalent to each firm having cost $\bar{\psi}\left(1-\kappa\right)$, where type

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[1] Just as in FL, the extension to the case of $N > 2$ firms is trivial.

[2] The construction can be easily extended to any absolutely continuous fixed cost distribution. Allowing for distributions with atoms would require the need for randomization for specific cost realizations, which goes against the spirit of the purification argument.

κ is uniformly distributed in $[0,1]$. Hence, if a firm with type $\kappa \in [0,1]$ ends up offering contract $c = (c_0, c_1) \in \mathbb{R}_+^2$ to an agent of type $t \in \{l, h\}$, its profits are given by

$$\Pi\left(c \mid t, \kappa\right) = \left(1 - p_t\right)\left(1 - c_1\right) - p_t c_0 - \bar{\psi}\left(1 - \kappa\right).$$

In what follows we construct a pure strategy equilibrium given by pure strategies

$$m : [0,1] \mapsto M,$$

where $m(\kappa)$ represents the menu offered by a firm with type κ. We focus on equilibria where only pairs of contracts are offered in equilibrium. Define as $u(\kappa) = (u_l(\kappa), u_h(\kappa))$ as the vector of utilities obtained by each type of consumer when offered the menu $m(\kappa)$. Based on the mixed equilibria construction in FL, we look for equilibria where on-path menu offers are ordered in terms of attractiveness, i.e., either

$$u(\kappa) \gg u(\kappa'), \text{ for any } 1 \geq \kappa > \kappa' \geq 0,$$

or

$$u(\kappa) \ll u(\kappa'), \text{ for any } 0 \leq \kappa < \kappa' \leq 1.$$

We refer to the first case as an increasing equilibrium and the second case as a decreasing equilibrium.

Define as $G_t(\cdot)$ the distribution of the utility obtained by an agent of type $t \in \{l, h\}$ when receiving offers from a single firm. Also define the probability of attracting an agent of any risk type as

$$G_0(u) \equiv G_l(u_l) + G_h(u_h).$$

For the remainder of this note, we assume that these three distributions are absolutely continuous. Also define $P_t(u)$ represents the maximal profits a firm can obtain from a consumer of type t, offering a contract that generates at least utility u_t to an agent of type t and at most utility $u_{t'}$ for an agent of type $t' \neq t$.

Our first result shows that any ordered equilibrium is necessarily increasing: firms with lower fixed cost make more attractive offers since they have a stronger (relative) preference towards attracting consumers.

Lemma 1. *(Ordering of offers in equilibrium) Any ordered equilibrium is necessarily an increasing equilibrium.*

Proof. Let's consider an arbitrary symmetric pure strategy equilibrium with offers $u(\cdot)$. A firm with type κ must not find it optimal to make offer generating profits $u(\kappa')$, for some $\kappa' \in (0, \kappa)$, which means that

$$G_l(u_l(\kappa))\left[P_l(u(\kappa)) - \bar{\psi}(1-\kappa)\right] + G_h(u_h(\kappa))\left[P_h(u(\kappa)) - \bar{\psi}(1-\kappa)\right] \geq$$
$$G_l(u_l(\kappa))\left[P_l(u(\kappa')) - \bar{\psi}(1-\kappa)\right] + G_h(u_h(\kappa))\left[P_h(u(\kappa')) - \bar{\psi}(1-\kappa)\right].$$

This equation can be written in the following way:

$$G_l(u_l(\kappa)) P_l(u(\kappa')) + G_h(u_h(\kappa)) P_h(u(\kappa')) - \left[G_l(u_l(\kappa)) P_l(u(\kappa)) + G_h(u_h(\kappa)) P_h(u(\kappa))\right] \leq$$
$$\left[G_0(u(\kappa')) - G_0(u(\kappa))\right]\kappa.$$

Using the reverse potential deviation, we have that

$$\left[G_0(u(\kappa')) - G_0(u(\kappa))\right]\bar{\psi}(1-\kappa) \geq$$
$$G_l(u_l(\kappa')) P_l(u(\kappa')) + G_h(u_h(\kappa')) P_h(u(\kappa')) - \left[G_l(u_l(\kappa)) P_l(u(\kappa)) + G_h(u_h(\kappa)) P_h(u(\kappa))\right] \geq$$
$$\left[G_0(u(\kappa')) - G_0(u(\kappa))\right]\bar{\psi}(1-\kappa').$$

This implies

$$G_0(u(\kappa')) < G_0(u(\kappa)).$$

Since the equilibrium offers are ordered, the only way this can be satisfied is by having

$$u_l(\kappa') < u_l(\kappa),$$

and

$$u_h(\kappa') < u_h(\kappa).$$

□

3 Necessary conditions

In this section we describe the necessary conditions for an increasing equilibrium. In section 5, we show that these conditions are also sufficient for the construction of a pure strategy equilibria. To focus on parameter values where a mixed equilibrium exists, we assume that cross-subsidization is optimal.

Assumption 1. *Cross-subsidization is optimal, which is equivalent to*

$$\mu_l \frac{\partial P_l\left(u^{RS}\right)}{\partial u_h} + \mu_h \frac{\partial P_h\left(u^{RS}\right)}{\partial u_h} > 0,$$

where u^{RS} is the utility vector generated by the RS menu offer.

In a decreasing equilibrium, the correspondence $u(\cdot) = (u_l(\cdot), u_h(\cdot))$ necessarily satisfies:

1. Local optimality of offers: firms must not find is optimal to marginally change the utility offered to low-risk agents, i.e.,

$$g_l(u_l(\kappa))\left[P_l(u(\kappa)) - \bar{\psi}(1-\kappa)\right] + G_l(u_l(\kappa))\frac{\partial P_l}{\partial u_l}(u(\kappa)) = 0.$$

2. Incentive constraints: a firm with cost κ should not find it profitable to make contract offer $u(\kappa')$, for any $\kappa' \neq \kappa$, which means that κ solves the problem

$$\Pi(\kappa) \equiv \max_{\kappa'} G_l(u_l(\kappa'))\left[P_l(u(\kappa')) - \bar{\psi}(1-\kappa)\right] + G_h(u_h(\kappa'))\left[P_h(u(\kappa')) - \bar{\psi}(1-\kappa)\right],$$

which implies that, using the first order conditions of this optimization problem together with local optimality condition above,

$$u_h'(\kappa)\begin{bmatrix} g_h(u_h(\kappa))\left[P_h(u(\kappa)) - \bar{\psi}(1-\kappa)\right] + \\ G_l(u_l(\kappa'))\frac{\partial P_l}{\partial u_h}(u(\kappa)) + G_h(u_h(\kappa'))\frac{\partial P_h}{\partial u_h}(u(\kappa')) \end{bmatrix} = 0.$$

3. Utility distribution: if the utility offers made by firms are increasing in their type $\kappa \in [0,1]$, it follows that the distribution of utilities offered by a single firm in a symmetric equilibrium satisfies

$$G_t(u_t(\kappa)) = \kappa,$$

$$G_0(u(\kappa)) = \kappa,$$

for any $\kappa \in [0,1]$ and $t \in \{l, h\}$.

4. Top optimality: insurance firms with lowest possible cost, having highest possible types, do not find it profitable to make even more attractive offers to high-risk agents, i.e.,

$$\mu_l \frac{\partial P_l(u(1))}{\partial u_h} + \mu_h \frac{\partial P_h(u(1))}{\partial u_h} = 0.$$

The incentive constraints described above imply the following envelope condition for the expected profits of each firm in equilibrium

$$\Pi'(\kappa) = G_0(\kappa)\bar{\psi} = \kappa\bar{\psi}.$$

This implies that the profits obtained by a firm with type κ, in equilibrium, is necessarily given by

$$\Pi(\kappa) = \int_0^\kappa s\bar{\psi}ds = \frac{\bar{\psi}\kappa^2}{2}.$$

Obviously firms with higher types, by having lower costs, are able to exploit their advantage in order to obtain positive profits in equilibrium. As a particular case of the expression above, we know that the expected profits obtained by a firm with type $\kappa = 0$ are

$$\Pi(1) = \mu_l P_l(u(1)) + \mu_l P_h(u(1)) = \frac{\bar{\psi}}{2}.$$

In summary, if $u : [0,1] \mapsto \Upsilon$ describes the utility offers made by firms in a monotone equilibrium then we necessarily have that final values $u(1)$ satisfy

$$\mu_l P_l(u(1)) + \mu_h P_h(u(1)) = \frac{\bar{\psi}}{2}, \tag{1}$$

$$\mu_l \frac{\partial P_l(u(1))}{\partial u_h} + \mu_h \frac{\partial P_h(u(1))}{\partial u_h} = 0. \tag{2}$$

Which come from necessary conditions 2, 3 and 4.

Additionally we have that, for $\kappa \in (0,1)$,

$$u_l'(\kappa) = \frac{P_l(u(\kappa)) - \bar{\psi}(1-\kappa)}{-\kappa \frac{\partial P_l}{\partial u_l}(u(\kappa))}, \tag{3}$$

$$u_h'(\kappa) = \frac{-\mu_h\left[P_h(u(\psi)) - \bar{\psi}(1-\kappa)\right]}{\kappa\left[\mu_l \frac{\partial P_l}{\partial u_h}(u(\psi)) + \mu_h \frac{\partial P_h}{\partial u_h}(u(\psi'))\right]}. \tag{4}$$

These equations come from combining necessary conditions 1, 2 and 3.

We denote this differential system (1)-(4) as $D(\bar{\psi})$. This is a non-linear non-autonomous differential system, which does not have a closed form solution. The study of this system is further complicated by the fact that the derivative function has non-continuous behavior around the point $(k, u_l, u_h) = \left(0, \underline{u}_l(\bar{\psi}), \underline{u}_h(\bar{\psi})\right)$. In the next section we explain the connection between this differential system and the mixed equilibria described in FL.

4 Connection to mixed equilibrium

The symmetric equilibria of the extreme case $\bar{\psi} = 0$ of homogeneous firms is completely characterized in FL.

The mixed equilibrium described in FL features menu offers that are indexed by the amount of cross subsidization between risk types $k \in \left[0, \bar{k}\right]$, which has distribution F. The utility offered to each risk type, for an offer with cross-subsidization level k is defined in the paper as

$$U(k) = \left(U_l(k), U_h(k)\right),$$

while the equilibrium menu offer delivering these utility levels is $\mathcal{M}^k$.

However, in the model with fixed costs, the notion of of cross-subsidization is not useful since firms with low costs actually have positive profits. For the sake of comparison of equilibria of different models, we define the following variable, which maps from percentiles of the distribution of cross-subsidization levels to utility offers:

$$u^0(\kappa) = \left(u_l^0(\kappa), u_h^0(\kappa)\right) : [0,1] \mapsto \mathbb{R}^2,$$

where

$$u_t^0(\kappa) = U_t\left(F^{-1}(\kappa)\right).$$

Notice that, since F is strictly increasing and continuous, its inverse is well-defined in $[0,1]$.

There is an obvious source of equilibrium multiplicity in the models considered due to the fact that many menus are consistent with delivering the same utility level to consumers (by adding, for example, irrelevant options). For this reason, the relevant equilibrium object is the distribution of the vector of utilities delivered by a single firm in the market. This object is denoted as $G \in \Delta\left(\mathbb{R}^2\right)$, which satisfies,[3]

$$G\left(u^0(\kappa)\right) = \kappa.$$

[3]Since offers are indexed by the amount of cross-subsidization between the two risk types, the two dimensions of this utility vector are perfectly correlated, i.e., for $\kappa_1, \kappa_2 \in [0,1]$, we have that

$$G\left(u_l^0(\kappa_1), u_l^0(\kappa_2)\right) = \min\{\kappa_1, \kappa_2\}.$$

Lemma 2. *The function* $u^0 : [0,1] \mapsto \Upsilon$ *solves* $D(0)$.

The existence of a well-behaved solution to this system for the case $\bar{\psi} = 0$ allows us to guarantee that the final condition of this differential system is also well-defined for small cost heterogeneity. The following lemma uses the continuous differentiability of the utility functions in order to insure existence of a final condition of the system $D(\bar{\psi})$.

Lemma 3. *(Final condition of ODE). For* $\bar{\psi} > 0$ *sufficiently small, equations* $(3) - (4)$ *have a solution* $u(1) = u^{\bar{\psi}}$ *satisfying* $u_l(1) > u_h(1)$ *and*

$$P_l(u_l(1)) > 0 > P_h(u_h(1)).$$

In the limiting case of homogeneous costs, we know that offers vary in the amount of cross-subsidization offered including all level between zero cross-subsidization and the efficient one. In the extended model constructed here the most attractive offers are not cross-subsidizing offers since insurance firms with low fixed costs use their cost-advantage in order to obtain positive expected profits. However, we can still show that firms with the highest possible fixed costs make offers with zero cross-subsidization, i.e., they will make zero profits regardless of the type of the agent. This offers differ, however, from the RS pair of contracts due to the presence of fixed costs.

Lemma 4. *Consider* $\bar{\psi}$ *sufficiently small. If* $u : [0,1] \mapsto \mathbb{R}^2$ *is an increasing solution to differential system* $(3) - (4)$ *with final condition* $u(1) = u^{\bar{\psi}}$ *, then*

$$P_t(u(0)) = \bar{\psi},$$

for $t \in \{l, h\}$.

The complexity of this differential system grants separate study in another project. For this reason, in this note, we take the smoothness of the solution of the system as given.

Assumption 2. *(Existence and smoothness of differential system) There exists* $\varepsilon_1 > 0$ *such that, for* $\bar{\psi} \in [0, \varepsilon_1]$*, the differential system system* $D(\bar{\psi})$ *has a unique solution, denoted as* $u^{\bar{\psi}} : [0,1] \mapsto \mathbb{R}^2$*, and that* $u^{\bar{\psi}}(\cdot)$ *is continuous in* ε.

5 Symmetric equilibrium

In this section we show that solutions to the differential system $D(\psi)$ generate equilibria of the studied model. We start by extending the supermodularity for expected profits presented in Lemma 5 in FL.

Fix risk type $t \in \{l, h\}$ and consider an arbitrary absolutely continuous utility distribution $G_t \in \Delta(\mathbb{R})$ with density g_t.[4] We define the expected utility from offering utility vector $u \in \Upsilon$ as, given fixed cost $\psi = \bar{\psi}(1-\kappa)$, as

$$\Pi(u;\kappa) \equiv G_l(u_l)\left[P_l(u) - \bar{\psi}(1-\kappa)\right] + G_h(u_h)\left[P_h(u) - \bar{\psi}(1-\kappa)\right].$$

As shown in Lemma 3 of FL, the function $P_t(\cdot)$ is continuously differentiable, which implies that function $\Pi(\cdot;\kappa)$ is differentiable. We define

$$M_t(u;\kappa) \equiv \frac{\partial \Pi(u;\kappa)}{\partial u_t}.$$

We now extend the supermodularity result presented in FL. As in Section 4 of FL, this supermodularity result is instrumental in showing that non-local deviations are not profitable to insurance firms.

Lemma 5. *(Pair-wise supermodularity) The function $\Pi(\cdot)$ is super-modular in any two of its arguments.*

Proof. For this proof suppose that $u_l > u_h$. The opposite case can be treated with a symmetric argument. From Lemma 3 in FL, we have that

$$\frac{\partial P_t(u)}{\partial u_t} < 0,$$

$$\frac{\partial P_t(u)}{\partial u_{t'}} \geq 0,$$

$$\frac{\partial^2 P_t(u)}{\partial u_t \partial u_{t'}} \geq 0,$$

with the second equation holding as an equality for $t = h$ and as a strict inequality for $t = l$ if $u_t > u_{t'}$. Direct derivation implies that

$$M_l(u;\kappa) = g_l(u_l)\left[P_l(u) - \bar{\psi}(1-\kappa)\right] + G_l(u_l)\frac{\partial P_l}{\partial u_l}(u).$$

This function is increasing κ since $\frac{\partial}{\partial \kappa} M_l(u;\kappa) = g_l(u_l)\bar{\psi}$. It is also increasing in u_h since $P_l(\cdot)$ is supermodular and increasing in u_h.

[4]The total mass of this distribution is normalized to $\mu_t \in (0,1)$ instead of 1, since an agent of type t can only be attracted to a firm if the realized risk type is t, which occurs with exogenous probability μ_t.

Now notice that

$$M_h(u;\psi) = g_h(u_h)\left[P_h(u) - \bar{\psi}(1-\kappa)\right] + G_l(u_l)\frac{\partial P_l(u)}{\partial u_h} + G_h(u_h)\frac{\partial P_h(u)}{\partial u_h}.$$

This function is increasing κ since $\frac{\partial}{\partial \kappa} M_h(u;\kappa) = g_h(u_h)\bar{\psi}$. It is also increasing in u_l since

$$\frac{\partial}{\partial u_l} M_h(u;\psi) = g_l(u_l)\frac{\partial P_l(u)}{\partial u_h} + G_l(u_l)\frac{\partial^2 P_l(u)}{\partial u_h \partial u_l} \geq 0.$$

□

Just as in FL, the monotonicity property described above is crucial to the analysis of potential off-path deviations by the insurance firms. Our next result uses the preliminary Lemma above in order to present an equilibrium existence and characterization result.

Proposition 1. *(Equilibrium characterization) Suppose that a monotonically decreasing solution of differential system $D(\psi)$ exists, then a symmetric equilibrium exists with firm strategies*

$$m^*(\psi) = \left\{\chi_l\left(u\left(\frac{\bar{\psi}-\psi}{\bar{\psi}}\right)\right), \chi_h\left(u\left(\frac{\bar{\psi}-\psi}{\bar{\psi}}\right)\right)\right\},$$

where χ_l, χ_h are defined as in FL.

Corollary 1. *(Purification) For $\varepsilon > 0$ sufficiently small, a symmetric pure strategy equilibrium of the model with costs $\psi \in \left[0, \bar{\psi}\right]$ exists. Moreover, the distribution of utility offers made by each firm converges, in weak convergence, to G (the mixed strategy utility distribution).*

6 Equilibrium properties

Pure strategy equilibria of the heterogeneous costs firms' model share several similarities with the mixed equilibria discussed in FL. Mainly, the uncertainty regarding the cost structure of each firms opponent's leads to uncertainty about how aggressive their offers will be. As mentioned in FL, this offer uncertainty precludes the simple cream-skimming deviation offers discussed in the literature and hence allow for the presence of cross-subsidies in equilibrium.

However, the introduction of cost heterogeneity also introduces new equilibrium features that better fit insurance markets. Mainly, equilibria do not require the use of randomization which is not observed in practice. More importantly, almost all firms in equilibrium enjoy positive expected profits due to their cost advantage relative to the lowest type in the market. As mentioned in section 3, in any monotone symmetric equilibrium the expected profits obtained by a firm of type $\kappa \in [0,1]$ is given by

$$\Pi(\kappa) = \frac{\bar{\psi}\kappa^2}{2}.$$

Hence, only firms with lowest type (i.e., highest fixed cost) obtain zero expected profits in equilibrium.

7 Concluding remarks

The goal of this note is to clarify the relationship between mixed equilibria of the competitive insurance model discussed in FL and pure strategy equilibria of the extended model with cost heterogeneity. Due to the non-linear preference structure, the construction of equilibria in this model is shown to be reduced to the solution of a system of non-linear equations. Given the complexity of this system, the analytical study of more complex models with cost heterogeneity seems difficult and further study might require the use of numerical exercises.

The cost structure considered in this paper is quite simplistic. In reality insurance firms differ in their internal administrative structure (which would affect fixed costs) as well as their provider network and deals (which would affect variable costs). A more realistic strategic model in this direction is crucial in order to fill the gap between theoretical work and structural estimation exercises in the literature.

8 Omitted proofs

Proof of Lemma 2. The distribution F satisfies

$$\frac{f(k)}{F(k)} = \frac{-\frac{\partial P_l(U(k))}{\partial u_l} U_l'(k)}{P_l(U(k))}, \tag{5}$$

and also $F\left(\bar{k}\right) = 1$.

Additionally, by construction, the mappings $k \mapsto U(k)$ satisfies, for all $k \in \left[0, \bar{k}\right]$,

$$\mu_l P_l(U(k)) + \mu_h P_h(U(k)) = 0. \tag{6}$$

Differentiating equation (6) and using the transformation

$$\left(u_t^0\right)'(\kappa) = \frac{U_t'\left(F^{-1}(\kappa)\right)}{f\left(F^{-1}(\kappa)\right)},$$

we have that

$$\frac{f\left(F^{-1}(\kappa)\right)}{\kappa} = \frac{-\frac{\partial P_l\left(u^0(\kappa)\right)}{\partial u_l}\left(u_l^0\right)'(\kappa) f\left(F^{-1}(\kappa)\right)}{P_l\left(u^0(\kappa)\right)},$$

$$\left[\begin{array}{c} \mu_l\left(\frac{\partial P_l\left(u^0(\kappa)\right)}{\partial u_l}\left(u_l^0\right)'(\kappa) f\left(F^{-1}(\kappa)\right) + \frac{\partial P_l\left(u^0(\kappa)\right)}{\partial u_h}\left(u_h^0\right)'(\kappa) f\left(F^{-1}(\kappa)\right)\right) \\ +\mu_h \frac{\partial P_h\left(u^0(\kappa)\right)}{\partial u_h}\left(u_h^0\right)'(\kappa) f\left(F^{-1}(\kappa)\right) \end{array}\right] = 0.$$

Rearranging this equations, and using the fact that $f(k) > 0$ for all $k \in \left(0, \bar{k}\right)$, we have

$$\left(u_l^0\right)'(\kappa) = \frac{P_l\left(u^0(\kappa)\right)}{-\frac{\partial P_l\left(u^0(\kappa)\right)}{\partial u_l}\kappa}. \tag{7}$$

which is exactly (3) and

$$\left(u_h^0\right)'(\kappa) = \frac{-\mu_l P_l\left(u^0(\kappa)\right)}{\left[\mu_h \frac{\partial P_h\left(u^0(\kappa)\right)}{\partial u_h} + \mu_l \frac{\partial P_l\left(u^0(\kappa)\right)}{\partial u_h}\right]}. \tag{8}$$

Using (7) and (6) in order to simplify (8) we have that

$$\left(u_h^0\right)'(\kappa) = \frac{-\mu_h P_h\left(u^0(\kappa)\right)}{\kappa\left[\mu_l \frac{\partial P_l}{\partial u_h}\left(u^0(\kappa)\right) + \mu_h \frac{\partial P_h}{\partial u_h}\left(u^0(\kappa)\right)\right]},$$

which is exactly (4). Finally, the final condition $u^0(1)$ satisfies (1), since cross-subsidizing offers generate zero profits, and (2), since $\bar{k}$ is the efficient level of cross-subsidization. □

Proof of Lemma 3. We are interested in finding a solution to the system of equations

$$H\left(u \mid \bar{\psi}\right) \equiv \left[\begin{array}{c} \mu_l P_l(u) + \mu_h P_h(u) - \frac{\bar{\psi}}{2} \\ \mu_l \frac{\partial P_l(u)}{\partial u_h} + \mu_h \frac{\partial P_h(u)}{\partial u_h} \end{array}\right] = 0.$$

This system has a solution u^* for $\bar{\psi} = 0$, which is the efficient cross-subsidizing offer described in FL and satisfies $u_l^* > u_h^*$. Also, we have that

$$\frac{d}{du}H\left(u\mid 0\right)=\left[\begin{array}{cc}\mu_l\frac{\partial P_l(u)}{\partial u_l} & 0\\ \mu_l\frac{\partial^2 P_l(u)}{\partial u_h \partial u_l} & \left(\mu_l\frac{\partial^2 P_l(u)}{\partial (u_h)^2}+\mu_h\frac{\partial^2 P_h(u)}{\partial (u_h)^2}\right)\end{array}\right],$$

has a strictly positive determinant since $P_l(\cdot)$ is strictly decreasing in u_l a and $P_t(\cdot)$ is strictly convex for $t \in \{l, h\}$. Using continuous differentiability of H, which follows from continuous differentiability of utility $u(\cdot)$, and the implicit function theorem we have that there exists $\varepsilon > 0$ and differentiable selection $u^\varepsilon : (0, \varepsilon) \mapsto \mathbb{R}^2$ satisfying

$$H\left(u^{\bar{\psi}} \mid \bar{\psi}\right)=0, \text{ for } \bar{\psi}\in(0,\varepsilon).$$

Continuity of the mapping u^ε implies the inequalities desired. □

We now show that the initial condition induced by the selection of this particular solution of the differential system is exactly the RS utility offer.

Proof of Lemma 4. Define $\bar{U}_K \equiv \{u \in \Upsilon \mid u_l - u_h \geq -K,\ P_h(u) \leq K \text{ and } P_l(u) \geq -K\}$. For K sufficiently small, this set is compact. Take $\bar{K} > 0$ to be such a value. From Assumption 2, we have that the solution $\{u(\kappa) \mid \kappa \in [0, 1]\}$ lies necessarily in $\bar{U}_{\bar{K}}$.

First, we show that $P_l(u(0)) = \bar{\psi}$. Assume by way of contradiction that $\left[P_l(u(\kappa)) - \bar{\psi}\right]$ is bounded away from zero for κ sufficiently small. From convexity and supermodularity of P_l we have that

$$\frac{\partial P_l\left(u\left(\kappa\right)\right)}{\partial u_l}<\frac{\partial P_l\left(u_h\left(1\right),u_l\left(0\right)\right)}{\partial u_l}<0.$$

It then follows necessarily that $u'(\cdot)$ is of the order $\frac{1}{\kappa}$ for κ small, which implies that

$$\lim_{\kappa\to 0} u_l\left(\kappa\right)=\pm\infty,$$

which is inconsistent with solution $u(\cdot)$ lying in a compact set. An analogous argument can be used to show that $P_h(u(0)) = \bar{\psi}$. The equalities

$$P_l\left(u\right)=P_h\left(u\right)=\bar{\psi},$$

have unique solution, which coincides with u^{RS} iff $\bar{\psi} = 0$. □

Finally, we present the proof of the main result in the note.

Proof of Proposition 1. We need to show that, for any $\psi \in \left[0, \bar{\psi}\right]$, offer $m^*(\psi)$ is optimal. We will index potential offers by the utility it delivers to each risk type, i.e., by $u \in \Upsilon$. It is easy to show that, within the set of menu offers that deliver utility vector u, the menu $\{\chi_l(u_l), \chi_h(u_h)\}$ achieves maximal profits. For the remainder of the proof let's fix $\kappa \in [0,1]$ and let $\psi = \bar{\psi}(1-\kappa)$, with on-path offer

$$u(\kappa) = (u_l(\kappa), u_h(\kappa)).$$

The set of on-path utilities offered to type $t \in \{l, h\}$ is

$$[u_t(0), u_t(1)].$$

We will divide the possible deviations in 7 cases:
A: $u = (u_l, u_h) \in [u_l(0), u_l(1)] \times [u_h(0), u_h(1)]$ with $u \neq u(\kappa)$;
B: $u = (u_l, u_h)$ such that $u_l < u_l(0)$;
C: $u = (u_l, u_h)$ such that $u_l \in [u_l(0), u_l(1)]$ and $u_h < u_h(0)$;
D: $u = (u_l, u_h)$ such that $u_l \in [u_l(0), u_l(1)]$ and $u_h(1) < u_h$;
E: $u = (u_l, u_h)$ such that $u_l(1) < u_l$ and $u_h \in [u_h(0), u_h(1)]$;
F: $u = (u_l, u_h)$ such that $u_t \notin [u_t(0), u_t(1)]$.

Case A: For concreteness assume that

$$u = (u_l, u_h) \in [u_l(0), u_l(1)] \times [u_h(0), u_h(1)],$$

and, for some κ_1, κ_2 satisfying $0 < \kappa_1 < \kappa < \kappa_2 < 1$,

$$u_l = u_l(\kappa_1) < u_l(\kappa),$$

$$u_h(\kappa) < u_h = u_h(\kappa_2).$$

We want to show that

$$\Pi(u; \kappa) - \Pi(u(\kappa); \kappa) \leq 0. \tag{9}$$

We will proceed by using the equality

$$\Pi(u; \kappa) - \Pi(u(\psi); \kappa) = \Delta_1 + \Delta_2,$$

where

$$\Delta_1 \equiv \Pi\left(u_l\left(\kappa_1\right), u_h\left(\kappa_2\right); \kappa\right) - \Pi\left(u_l\left(\kappa_1\right), u_h\left(\kappa\right); \kappa\right),$$

$$\Delta_2 \equiv \Pi\left(u_l\left(\kappa_1\right), u_h\left(\kappa\right); \kappa\right) - \Pi\left(u\left(\kappa\right); \kappa\right).$$

However we have that

$$\begin{aligned}
\Delta_1 &= \int_{\kappa}^{\kappa_2} M_h\left(u_l\left(\kappa_1\right), u_h\left(s\right); \kappa\right) u_h'\left(s\right) ds \\
&\leq \int_{\kappa}^{\kappa_2} M_h\left(u_l\left(s\right), u_h\left(s\right); s\right) u_h'\left(s\right) ds = 0,
\end{aligned}$$

which follows from supermodularity Lemma 5.

Also notice that, from Lemma 5,

$$\begin{aligned}
\Delta_2 &= \int_{\kappa_1}^{\kappa} M_l\left(u_l\left(s\right), u_h\left(\kappa\right), \kappa\right) u_l'\left(s\right) ds \\
&\leq \int_{\kappa_1}^{\kappa} M_l\left(u_l\left(s\right), u_h\left(s\right), s\right) u_l'\left(s\right) ds \leq 0.
\end{aligned}$$

This implies equation (9). This argument can be extended to all possible deviations in Case A.

Case B: Any such offer is equivalent to offer $u' \equiv \left(u_l\left(0\right), u_h\right)$, which is in Case A and hence is not profitable.

Case C: Any such offer is weakly dominated offer $u' \equiv \left(u_l, u_h\left(0\right)\right)$, which is in Case A and hence is not profitable.

Case D: Let's consider deviation by firm of type $\kappa \in [0,1]$ to alternative offer $\left(u_l\left(\kappa'\right), u_h'\right)$ such that $u_h' > u_h\left(1\right)$. We then have that

$$\Pi\left(u_l\left(\kappa'\right), u_h'; \kappa\right) - \Pi\left(u_l\left(\kappa\right), u_h\left(\kappa\right); \kappa\right) = \begin{array}{c} \overbrace{\Pi\left(u_l\left(\kappa'\right), u_h'; \kappa\right) - \Pi\left(u_l\left(\kappa'\right), u_h\left(1\right); \kappa\right)}^{\equiv\Delta_1} \\ + \underbrace{\Pi\left(u_l\left(\kappa\right), u_h\left(1\right); \kappa\right) - \Pi\left(u_l\left(\kappa\right), u_h\left(\kappa\right); \kappa\right)}_{\Delta_2} \end{array}.$$

But notice that, by pairwise supermodularity

$$\begin{aligned}
\Delta_1 &\leq \Pi\left(u_l\left(1\right), u_h'; 1\right) - \Pi\left(u_l\left(1\right), u_h\left(1\right); 1\right) \\
&= \int_{u_h(1)}^{u_h'} \mu_l \frac{\partial P_l\left(u_l\left(1\right), s\right)}{\partial u_h} + \mu_h \frac{\partial P_h\left(u_l\left(1\right), s\right)}{\partial u_h} ds
\end{aligned}$$

which is negative since $P_t(\cdot)$ is convex, for each $t \in \{l, h\}$, and condition (2).

Case E: Any such offer is weakly dominated offer $u' \equiv (u_l(1), u_h)$, which is in Case A and hence is not profitable.

Case F: Any such offer is dominated by an offer in Case A. The proof is a straightforward combination of the arguments for cases B-E. $\square$

References

Farinha Luz, V.: Characterization and uniqueness of equilibrium in competitive insurance, *Working paper*.