Robust Selling Mechanisms*

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Abstract

We consider the problem of a seller who faces a privately informed buyer and only knows one moment of the distribution from which values are drawn. In face of this uncertainty, the seller maximizes his worst-case expected profits. We show that a robustness property of the optimal mechanism imposes restrictions on the seller's ex-post profit function. These restrictions are used to derive the optimal mechanism. The optimal mechanism entails distortions at the intensive margin, e.g., except for the highest value buyer, sales will take place with probability strictly smaller than one. The seller can implement such allocation by committing to post prices drawn from a non-degenerate distribution, so that randomizing over prices is an optimal robust selling mechanism. We extend the model to deal with the case in which: (i) M goods are sold and the buyer's private information is multidimensional and (ii) the seller and the buyer interact for several periods. In the case of multiple goods, there are several optimal mechanisms. With multiple goods full bundling is optimal, as well as selling the goods in a fully separable way. In the dynamic model, we show that repetition, period by period, of the static-optimal mechanism is optimal.

Keywords: Robust Mechanism Design, Monopolistic Pricing under Uncertainty, Price Discrimination

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Contents

1 Introduction

This paper considers the problem faced by a monopolist (he), who – except for knowing one moment of the distribution – is unaware of the distribution from which the consumer's (she) value is drawn. In face of such ambiguity, he seeks to design a mechanism that is robust, in the sense of maximizing his worst-case expected profits over all distributions satisfying the moment condition. In its more general version, our model considers the case in which $M > 1$ goods are sold (therefore, valuation is multidimensional) and the monopolist and the consumer interact for multiple periods. Therefore, the problem we tackle is one of robust multidimensional dynamic pricing.

For most of the exposition we consider a seller that is informed about the first moment of the type distribution. The assumption that the seller knows at least the first moment of the distribution of values makes the problem interesting. Under full ignorance of the distribution, the optimal mechanism would be trivial: set a price equal to the lowest value the consumer might attribute to the good. Moreover, it might well be the case that the seller, proceeding as an econometrician as proposed by Segal (2003), has access to enough data to estimate (say, through an hedonic regression) the mean of the distribution of values, but his sample falls short of data for a non-parametric (consistent) estimation of the whole distribution. Alternatively, it might be common knowledge that consumer expects her value to be $k > 0$ but, before purchase and after the seller offers her a mechanism, might acquire relevant information about her value. In such interpretation, due to Gabriel Carroll, if the seller is uncertain about the agent's information acquisition technology, all he will know is that values are drawn from a distribution with mean k.

We start, to fix ideas, by considering a simple one-period case with single-dimensional private information. Since the seminal paper Myerson (1981), much is known about such case when the consumer has a unit demand, and the monopolist knows the distribution F from which this value is drawn. Bulow and Roberts (1989) demonstrate that the optimal selling mechanism resembles the solution of a standard monopolist's problem. In fact, for a given price p, if one defines "quantities" sold as the probability of sales under F – namely, $1 - F(p)$ – total revenues can be written as $p.(1 - F(p))$. A monopolist would then compute marginal revenues, $p - \frac{1-F(p)}{f(p)}$, and sell if and only if they are larger than costs. The rule coincides with the one derived by Myerson (1981). Moreover, when marginal revenues are monotone, the seller can implement such optimal mechanism by posting a single price.

Although quite simple, the solution requires full knowledge of "demand" – the distribution F – and cannot be pursued if the monopolist only knows the expected value (denote it by $k > 0$) of the consumer's valuation. Solving for the mechanism that maximizes expected profits under the worst case distribution involves a couple of steps. First of all, it is convenient to think of the robust mechanism design problem as being the seller's best response to an adversary nature, who seeks to minimize his expected profits by choice of distributions. This minimization problem is constrained by the fact that the distribution must integrate to one, and, by assumption, its expected value is k. Once one incorporates those constraints in a Lagrangian functional, nature minimizes, by choice of distributions, the expected value of the seller's profits subtracted by the inner product of the constraints and their shadow costs (the Lagrangian multipliers).

In our first result, we show that nature will only place positive likelihood on values $\theta \in [0, 1]$ for which the seller's profits equal $\xi\theta - \lambda$, where ξ is the shadow cost of the constraint that imposes that the average of the distribution must be k, and λ is the shadow cost of the constraint that the distribution must integrate to one. As a corollary of such result, we show that seller's profits are piecewise linear: they are equal to zero over the region in which there are no sales, and is linear, with slope ξ, when the mechanism entails sales with positive probability. It follows that the robust mechanism imposes restrictions on the seller's payoff levels. One interpretation is that, knowing only the first moment of the distribution of values, the seller can only explore linearly higher consumer's values; else nature could move likelihood weights in a way that preserves the shadow costs imposed by its restrictions and reduces the seller's expected profits. Alternatively, under the information acquisition interpretation of the model, the stage at which the consumer obtains new information adds volatility to the seller's (degenerate) prior. A profit function with a call option format is then optimal.

Having established that profits are linear in values conditional on sales, finding the robust allocation is simple. Indeed, by imposing that the derivative of the profit function with respect to values is ξ conditional on sales, we obtain an ordinary differential equation whose solution yields the robust sales' decision. As opposed to what prevails in the standard Bayesian problem, in the robust mechanism sales take place with probability smaller than one for consumers with valuation below the highest one. Put differently, there are distortions at the intensive margin. The interpretation is simple. To insure against uncertainty, the seller finds desirable to sell to a marginal consumer with a low value $\theta' < 1$. Nevertheless, if he were to sell with probability one to this marginal consumer, the maximum he could charge from infra marginal consumers (those with values $\theta > \theta'$). By distorting the consumption of lower types, the seller can charge more from higher types: a standard discriminatory practice, although with a very different rationale from standard price discrimination theories.[1] Price discrimination is the way by which the seller simultaneously insures against uncertainty and charges high prices from infra marginal consumers.

[1]In particular, in a setting like ours, if the seller is a standard expected utility maximizer, he will only find it optimal to discriminate if either the consumer's payoff or his cost function have some curvature.

There are two different ways to indirectly implement such robust allocation. Since the robust allocation displays distortions at the intensive margin, we can use standard non-linear prices to implement it. The second, and more interesting, way to implement the robust allocation explicitly uses the lack of curvature in the consumer's payoff. In fact, the seller can implement the robust allocation by committing to pick a price from a well designed, non-degenerate, probability distribution of prices. Therefore, randomizing over posted prices is an optimal robust mechanism.

We extend the simple model in three directions. First, we allow for the seller to have knowledge of an arbitrary moment condition of the distribution of values. As we shall see, the main features just described remain qualitatively the same; in particular, the robust allocation entails distortions at the intensive margin. Second, we consider the case in which the monopolists sells M goods and all he knows is that the consumer has unit demand for each of the goods, and that her (multidimensional) value lies in $[0,1]^M$ and has expected value of $k = (k, ..., k) \in [0,1]^M$. We assume that the monopolist faces no technological constraints and can produce up to one unit of each good. In Bayesian settings, not much is known about the solution of such a problem and even standard features of single-good sales mechanisms, such as monotonicity of profits in consumer's values, do not extend in general to multidimensional profit maximizing mechanisms (see Hart and Reny (forthcoming)).

In contrast, a general characterization of the robust multidimensional mechanism is not only feasible, but follows from the same arguments used in the single-dimensional problem. In fact, we consider the problem faced by nature in minimizing, by choice of distributions, the seller's expected profits subject to the constraints that the worst case distribution has to integrate to one and lead to an expected value of k. Much as in the one good case, we establish that the seller's profit function has also a lower piecewise linear envelope in the multidimensional case, and the nature puts positive weight exactly where these functions coincide. Differentiation of this revenue function leads to a system of partial differential equations that corresponds to a set of necessary conditions that a robust scheme must satisfy. Perhaps surprisingly, two completely different mechanisms satisfy those necessary conditions: one in which the monopolist sells the goods in a fully separable fashion (proceeding as if he was dealing with M separate consumers with unit demands for each of the goods he sells), and the other in which the seller constructs an "aggregate" value (by summing the consumer's values for all goods) and sells the goods in fixed proportions in a bundle. We then verify that both mechanisms are indeed optimal by: (i) showing that, coupled with a properly chosen distribution, the mechanism with fully separate sales is part of a Nash equilibrium of the zero sum game played by nature and the seller; and (ii) proving that value the full bundling mechanism yields is the same as the fully separate sale.

Third, we consider the case in which the consumer and the monopolist interact for multiple periods and the consumer's values might evolve over time. There are two main reasons to consider the role of dynamics in our model. Even if the seller has little information about the buyer's value, extra information might become available in future periods, leading to better revenue extraction. However ambiguity on the side of the seller regarding this additional information makes the result non-trivial. In our dynamic model this is incorporated by looking at pricing rules that potentially depend on previous consumption behavior by the buyer. Also, knowledge of average valuations is naturally connected to learning and information acquisition, as conditional expectations with increasing information sets follow a martingale. However, information acquisition is a dynamic phenomenon that should be discussed in a fully dynamic model.

A central part of our analysis deals with the key distinction between the case of multiple goods and multiple periods: the sequential revelation of information. A consequence of the sequential revelation of information is a failure of the revelation principle in the presence of ambiguity. One methodological contribution of this paper is to present a way of dealing with dynamics in a world with ambiguity. The seller chooses from the larger set of indirect mechanisms and, in the face of large ambiguity, considers all possible type distribution as well the whole set of optimal reporting strategies induced by each distribution. Our main result is the irrelevance of dynamics in the presence of large ambiguity, i.e., repeated static optimal pricing is optimal.

We next put all those pieces together and show that the optimal dynamic multidimensional robust mechanism involves the repetition, period by period, of the static multidimensional mechanism, which, in turn, can either treat each good in a fully separable way, or sell them in proportional bundles as a function of aggregate valuations.

Related literature

Our paper is part of a growing literature on mechanism design with principals with maximin preferences.[2] Frankel (2014) and Carrasco and Moreira (2013) consider decision-making problems with non-transferable utility in which a maximin principal is unaware of the agent's bias (in Frankel (2014)) or the distribution of states (Carrasco and Moreira (2013)). In an otherwise standard procurement setting à la Laffont and Tirole (1986), Garrett (2014) considers the case of a principal who does not know the producer's disutility of effort, and show that a simple fixed-price-cost-reimbursement (FPCR) menu minimizes the principal's maximum expected payment to the agent. In Carroll (2015), the principal only partially knows the set of actions available to the agent; he shows that if the principal maximizes expected profits under worst case set of actions, the optimal contract is linear in output. Our work differs from those listed above by considering a seller's pricing decision.

[2] There is also a growing literature with maximin agents. Bose et al. (2006) and Wolitsky (2014) are examples of analysis of, respectively, optimal auctions and bilateral trade when agents have maximin preferences.

There are, nevertheless, a set of papers that focus on pricing with unknown distribution of values and posit that the seller has preferences for robustness.[3] Bergemann and Schlag (2008, 2011) are perhaps the first to do so. In their first paper, they consider the case in which the seller designs a mechanism to minimize the maximum regret, whereas in the second they also consider a maximin procedure. In both cases, they work within a static and single-dimensional case. Similar to what we find, they show that randomizing over prices is a way to insure against uncertainty in the minimax problem. Pointing out that, without further restrictions, the maximin problem entails the trivial solution of charging the lowest possible valuation with probability one, they consider "local robustness", that is, maximin pricing over neighborhoods around a given distribution. They show that, starting at the certainty case, the charged price decreases as uncertainty (as measured by the size of the neighborhood) increases. Instead of considering a minimax criterion or working with neighborhoods around a given distribution, to avoid a trivial solution for the maximin problem, we assume that a given moment of the distribution of types is known. This complements their minimax analysis, on the one hand, and allows for an analysis of the multidimensional and repeated cases, on the other.

In independent work, Carroll (2013) considers a setting in which a buyer knows the expected value of its willingness to pay, but can acquire information about it before purchasing the good. The seller, who only knows the prior from which the expected valuation is drawn, designs a mechanism to maximize the worst case (over information acquisition technologies, which amounts to choosing among mean preserving spreads of the prior) expected profits.[4] Our single-good case corresponds to a special case of his when the seller's prior is degenerate. For the case of a single good, however, we also consider the case of knowledge of an arbitrary moment by the seller. We also extend the analysis to $M > 1$ goods (and multidimensional values) and repeated sales.

Handel and Misra (forthcoming) consider the problem of a monopolist who launches a new product and, without knowledge of (the time invariant) demand, decides – restricting attention to price posting mechanisms – on intertemporal prices to minimize maximum intertemporal regret. They show that prices decrease over time if consumers are homogenous, and increase if consumers are heterogeneous. Caldentey et al. (2015) also consider minimax intertemporal pricing for the case in which seller restrict attention to posting price mechanisms. However, on top of not knowing demand, the seller does not know the arrival process of consumers in their paper. They also establish that optimal price paths are decreasing when buyers are rational. In

[3]Segal (2003) also considered the case in which a monopolist did not know the distribution from which values were drawn. Rather than positing that the seller has preferences for robustness, he considered a seller who proceeds as an econometrician and estimates, from the mechanisms offered to subset of consumers, the distribution of values.

[4]In the introduction, we have borrowed Carroll's story to justify why the seller might know the mean of the distribution from which values are drawn.

contrast to those papers, we consider maximin design (with the restriction that expected values follows martingale[5]) and allow for general mechanisms. We show that the optimal dynamic mechanism is time-invariant. Such time-invariance result can be related to the recent literature on (Bayesian) dynamic mechanism design. In particular, Pavan et al. (2015) show that the amount of informational rents that must be left to an agent whose private information follows an AR(1) process depends on its the degree of persistence. It should then be no surprise that the adversarial nature the seller faces chooses consumer's types to be fully persistent in our model.

To the best of our knowledge, we are the first to consider multidimensional robust design. Not much is known in general for Bayesian multidimensional design (see, for instance, Hart and Reny (forthcoming) and references therein). By looking at worst-case selling procedures, we are able to fully derive optimal mechanisms and show that they involve either full separation or full bundling (in contrast, for example, to the mixed-bundling solution of McAfee et al. (1989) and the literature that followed).

Organization

Section 2 lays down the general model. In Section 3, we derive the optimal robust mechanism for the static setting in which the monopolist sells one good. We tackle the robust design problem using two different approaches. In the first one, we use standard Lagrangian technique, whereas, in the second, we recast the robust design problem in terms of a zero-sum game played by the monopolist and an adversary nature who chooses distribution to minimize his expected profits. This latter approach proves useful to derive the optimal robust mechanism for an arbitrary moment condition that might be known by the monopolist. In Section 4 we consider the robust design for the case in which the monopolist sells M goods. We move to the case of a $T-$period robust design in Section 5. We draw our concluding remarks in Section 6.

2 Model

A monopolist (or seller) can produce $M \geq 1$ indivisible and non-storable goods at zero cost in each period $t \in \{1,..,T\}$, $T \geq 1$. The seller faces a consumer who has valuation for the good in period t denoted as $\theta_t = \left(\theta^1, \ldots, \theta^M\right) \in [0,1]^M$. A sequence of valuations is denoted by $\theta^t = (\theta_1, ..., \theta_t) \in [0,1]^{Mt}$, for any $t \leq T$. If quantity $\mathbf{q}_t \in [0,1]^M$ (we use bold to represent the vector of quantities) and transfers p_t are made in each period $t \in T$, the utility obtained by the

[5] Again, this can be justified by an information acquisition story, since, if information is acquired over time, at any given time the martingale property must hold.

consumer is given by

$$\sum_{t=1}^{T} \delta^{t-1} \left(\mathbf{q}_t \cdot \theta_t - p_t \right),$$

and the seller's profits are

$$\sum_{t=1}^{T} \delta^{t-1} p_t.$$

The set of direct mechanisms is defined as

$$\mathcal{M} \equiv \left\{ m = (\mathbf{q}_t, p_t); (\mathbf{q}_t, p_t) : [0,1]^{MT} \to [0,1] \times \mathbb{R} \text{ is } \theta^t\text{-measurable} \right\}.$$

The set of mechanisms with arbitrary message spaces is given by $\overline{\mathcal{M}} \supsetneq \mathcal{M}$. For any mechanism m with message space A in each period, a reporting strategy is $\sigma = (\sigma_t)_{t=1}^{T}$ with function $\sigma_t : [0,1]^{MT} \to A$ which is measurable with respect to $\theta^t \in [0,1]^{Mt}$. The set of all reporting strategies is Σ_m and, for any direct mechanism, the truth-telling strategy is denoted as σ^{TT}.[6] The realized payoff of an agent following strategy $\sigma \in \Sigma_m$ in mechanism $m \in \overline{\mathcal{M}}$ is

$$\mathcal{U}_m \left(\sigma \mid \theta^T \right) \equiv \sum_{t=1}^{T} \delta^{t-1} \left[\mathbf{q}_t \left(\sigma \left(\theta^T \right) \right) \cdot \theta_t - p_t \left(\sigma \left(\theta^T \right) \right) \right],$$

and the realized firm profits are given by

$$\Pi_m \left(\sigma \mid \theta^T \right) \equiv \sum_{t=1}^{T} \delta^{t-1} p_t \left(\sigma \left(\theta^T \right) \right).$$

When dealing with a direct mechanism we will also use $\mathcal{U}_m \left(\theta^T \right)$ and $\Pi_m \left(\theta^T \right)$ to denote realized payoffs for the buyer (the rent function) and the seller (the profit function) under truth-telling.

We assume that the type distribution $F \in \Delta \left([0,1]^{MT} \right)$ is known by the buyers but unknown by the seller. The seller only knows that the set of possible distributions is $\mathcal{F} \subseteq \Delta \left([0,1]^{MT} \right)$ but does not have a probability distribution over this set. Instead, the seller is ambiguity averse. The seller's problem amounts to designing a mechanism to maximize his worst-case expected profits over all distributions in $\mathcal{F}$. In this paper the defining property of this set it assumed to be a set of moment conditions, which is the only market information available to the seller. The definition of this set is presented for each case of interest studied. Over the next sections, to

[6] We assume that the set of allocations that are feasible for a buyer is compact, i.e., $\{ (q^\infty, p^\infty) ; \exists a^\infty \in A^\infty \text{ such that } (q_t (a^\infty), p_t (a^\infty)) = (q_t^\infty, p_t^\infty), \forall t \in T \}$ is compact in the product topology. This is required to guarantee the buyer always has an optimal reporting strategy.

build the ideas that will allow us to find the robust mechanism for the general model we just laid out, we will consider the seller's problem for some special cases of interest.

3 One period and one good

We start focusing on the static case where the seller holds one divisible good and only has information about the average of the type distribution. Formally, the set of possible distributions is

$$\mathcal{F} \equiv \left\{ F \in \Delta\left([0,1]\right); \int \theta dF\left(\theta\right) = k \right\},$$

for some $k \in (0,1)$.

The seller wants to maximize the revenue guarantee given by this moment condition alone. This is done by considering the worst-case expected profits from the set of possible distributions. Since the buyer has complete information at announcement stage, the set of optimal reporting strategies is independent of the actual type distribution $F \in \mathcal{F}$. As a consequence, any selection from this set that is independent of the distribution choice can be implemented through a direct mechanism with truthful strategies. Hence, we restrict attention to incentive compatible direct mechanisms. The seller's problem becomes:

$$\max_{m\in\mathcal{M}} \min_{F\in\mathcal{F}} \int_0^1 p\left(\theta\right) dF\left(\theta\right), \tag{1}$$

subject to participation constraint

$$\mathcal{U}_m\left(\theta\right) \geq 0,$$

and incentive compatibility constraints

$$\mathcal{U}_m\left(\theta\right) \geq \theta q\left(\widehat{\theta}\right) - p\left(\widehat{\theta}\right),$$

for all $\theta, \hat{\theta} \in [0,1]$.

As usual under the single crossing condition, incentive compatibility is equivalent to the envelope condition:

$$\mathcal{U}_m\left(\theta\right) = \mathcal{U}_m\left(0\right) + \int_0^\theta q\left(\tau\right) d\tau, \tag{2}$$

and the monotonicity condition: $q\left(\cdot\right)$ is non-decreasing. Substituting equation (2) in the objective and noticing that, regardless of worst-case distribution, the seller will always pick a mechanism with $\mathcal{U}_m(0) = 0$, his problem can be equivalently rewritten as:

$$\max_{m \in \mathcal{M}} \min_{F \in \mathcal{F}} \int_0^1 \Pi_m(\theta) dF(\theta)$$

subject to $q(\theta)$ non-decreasing and

$$\Pi_m(\theta) := p(\theta) = \theta q(\theta) - \int_0^\theta q(\tau)\, d\tau$$

is the profit function associated to mechanism $m = (q, p)$.

We follow the classical approach in looking at the relaxed problem without monotonicity constraints and showing that the solution to this problem indeed satisfies the ignored constraints.

3.1 Worst case payoff

We start by fixing an arbitrary incentive compatible mechanism and the associated profit function $\Pi(\theta)$, i.e., any function $\Pi(\theta)$ that satisfies $\Pi(0) = 0$ and $\Pi(\theta)$ is non-decreasing[7]. Because of this latter property, it is without loss of generality to assume that $\Pi(\theta)$ is left-continuous.

Let us relax the constraints which require that the cumulative distributions are probability distributions that have mean k and consider the following relaxed problem of finding the worst expected profits:

$$\min_{F \in \mathcal{D}} \int \Pi(\theta)\, dF(\theta) \tag{3}$$

subject to aggregate mass being smaller than one

$$\int dF(\theta) \leq 1,$$

and the average condition

$$k - \int \theta dF(\theta) \leq 0,$$

where $\mathcal{D} = \{F : [0,1] \to \mathbb{R}_+$ is non-decreasing, right continuous and bounded function$\}$. This,

[7]It follows that, whenever it exists, the derivative of the profit function $\Pi(\theta) = \theta q(\theta) - \int_0^\theta q(\tau)\, d\tau$ associated to the incentive compatibility allocation $q(\cdot)$ is a.e. $\Pi'(\theta) = q(\theta) + \theta q'(\theta) - q(\theta) = \theta q'(\theta) \geq 0$. Therefore, $\Pi'(\theta) \geq 0$ a.e. if and only $q'(\theta) \geq 0$ a.e.

in turn, allows us to show:

Lemma 3.1. *(Existence) There is a solution to the problem (3). At the robust mechanism, the solution belongs to $\mathcal{F}$.*

Proof of Lemma 3.1. The set $\mathcal{D}$ is compact and the constraints of problem (3) are closed with respect to the weak topology. It is straightforward to see that the objective function is also lower semi-continuous with respect to the weak topology ($\Pi(\theta)$ is a left-continuous and non-decreasing function). The second constraint must bind. Otherwise, using the Lagrangian approach presented below, we know that if $\xi^* = 0$, then the distribution that attains the minimum would be concentrated at $\theta = 0$ since $\Pi(\theta) + \lambda^*$ is a positive function for $\theta \in (0, 1]$. However, this violates the second constraint (unless $k = q = \lambda^* = 0$, in which case the Dirac measure concentrated at $\theta = 0$ is the optimal distribution). Hence, $\xi^* > 0$ and the second constraint being binding. In the proof of Proposition 3.1 we argue that if $\xi^* > 0$, then $\lambda^* > 0$ and, consequently, the first constraint should bind at the robust mechanism. □

Let $F^* \in \mathcal{F}$ be a solution of the problem in (3). Standard arguments (see, for instance, Luenberger (1969)) imply that there exists a Lagrangian functional $L : \mathcal{D} \times \mathbb{R}^2_+ \to \mathbb{R}$ of problem (3) defined by

$$L(F; \lambda, \xi) = \int \Pi(\theta) dF(\theta) + \lambda \left(\int dF(\theta) - 1 \right) + \xi \left(k - \int \theta dF(\theta) \right)$$

so that the worst case distribution F^* and multipliers $\lambda^* \geq 0$ and $\xi^* > 0$ satisfy the saddle point condition:

$$L(F^*; \lambda, \xi) \leq L(F^*; \lambda^*, \xi^*) \leq L(F; \lambda^*, \xi^*),$$

for all $(F, \lambda, \xi) \in \mathcal{D} \times \mathbb{R}^2_+$. Hence, ignoring constant terms,

$$\int \Pi(\theta)\, dF^*(\theta) = \min_{F \in \mathcal{D}} \int \left[\Pi(\theta) + \lambda^* - \xi^* \theta\right] dF(\theta).$$

Clearly, problem (3) is well defined only if $\Pi(\theta) + \lambda^* - \xi^*\theta \geq 0$, for all $\theta \in [0, 1]$. In fact, if for some $\theta \in [0, 1)$ such that $\Pi(\theta) + \lambda^* - \xi^*\theta < 0$, one would have

$$L(N.H_\theta; \lambda^*, \xi^*) \to -\infty \text{ when } N \to \infty,$$

where H_θ is the Heaviside function at θ, i.e., $H_\theta(x) = 0$ for $x \in [0, \theta]$ and $H_\theta(x) = 1$ for $x \in (\theta, 1]$. Since $\Pi(\theta)$ is left-continuous and non-decreasing, then $\Pi(1) + \lambda^* - \xi \geq 0$. We can then define the sets of types where the profit function is the affine envelope function $\xi^*\theta - \lambda^*$, $I = \{\theta \in [0, 1]; \Pi(\theta) = \xi^*\theta - \lambda^*\}$, and its complement $I^c = \{\theta \in [0, 1]; \Pi(\theta) > \xi^*\theta - \lambda^*\}$. At the optimal distribution F^*, the mass of types where the profit is above the affine envelope must be zero, i.e., $dF^*(I^c) = 0$. Otherwise, by moving weight from I^c to I, the objective of the minimization problem could be improved by reducing expected profit.

More interestingly, at the robust mechanism the optimal distribution must spread mass for all types above a certain cutoff type. The intuition is that the "nature",[8] who is minimizing the objective function, has zero shadow profit $\Pi(\theta) + \lambda^* - \xi^*\theta$ for all types in I and puts zero weight for types in I^c, which have positive shadow profit. Therefore, if a top interval of types, say $[\bar{\theta}, 1]$, have positive positive shadow profit, the seller could reduce profit for those types such that the shadow profit is slightly below zero but still greater than the profit obtained with types below $\bar{\theta}$ (once the profit function is non-decreasing). The nature would then react putting all mass in this top interval and the seller would ensure better expected profit.

The next proposition formally proves that, at the robust mechanism, $I = [\theta^*, 1]$, where $\xi^*\theta^* = \lambda^*$. However, let us first state a lemma which will be important in the proof of the next proposition. In particular, it implies that piecewise linear lower envelope of a profit function is also a profit function associates to an allocation whose maximum quantity is not greater than the allocation that generates the original profit function. That is, if the original profit function satisfies the capacity constraint, the lower envelope piecewise linear also does.

Lemma 3.2. *(Auxiliary) Let $q(\cdot)$ be any allocation such that $0 \leq q(\theta) \leq 1$ and for some $\xi^* > 0$ and $\lambda^* \geq 0$*

$$\Pi(\theta) := \theta q(\theta) - \int_0^\theta q(\tau)d\tau \geq \xi^*\theta - \lambda^*,$$

for all $\theta \in [0, 1]$. Then,

$$q(\theta) \geq \frac{1}{\theta^*}\int_0^{\theta^*} q(\tau)d\tau + \xi^* \; ln \; \left(\frac{\theta}{\theta^*}\right),$$

for all $\theta \in [0, 1]$, where $\xi^*\theta^* = \lambda^*$.

Proof of Lemma 3.2. Defining $\psi(\theta) = \int_0^\theta q(\tau)d\tau$, the hypothesis of the lemma is equivalent to $\theta\psi'(\theta) - \psi(\theta) \geq \xi^*(\theta - \theta^*)$, for all $\theta \in [0, 1]$. Now, notice that

$$\begin{aligned} \tfrac{\psi(\theta)}{\theta} - \tfrac{\psi(\theta^*)}{\theta^*} - \lambda^*\left(\tfrac{1}{\theta} - \tfrac{1}{\theta^*}\right) &= \int_{\theta^*}^\theta \left(\tfrac{\psi(\tau)}{\tau}\right)' d\tau + \lambda^* \int_{\theta^*}^\theta \tfrac{1}{\tau^2} d\tau \\ &= \int_{\theta^*}^\theta \tfrac{\tau\psi'(\tau) - \psi(\tau)}{\tau^2} d\tau + \lambda^* \int_{\theta^*}^\theta \tfrac{1}{\tau^2} d\tau \\ &\geq \int_{\theta^*}^\theta \tfrac{\xi^*(\tau - \theta^*) + \lambda^*}{\tau^2} d\tau = \xi^* \ln\left(\tfrac{\theta}{\theta^*}\right). \end{aligned}$$

[8]In subsection 3.3 we will explore this interpretation to derive that the robust mechanism as a part of a Nash equilibrium between the seller and the adversary nature in zero-sum game.

Hence,

$$\begin{aligned} q(\theta) & \geq \tfrac{\psi(\theta)}{\theta}+\xi^*\left(1-\tfrac{\theta^*}{\theta}\right) \\ & \geq \tfrac{\psi(\theta^*)}{\theta^*}+\xi^* \ln\ \left(\tfrac{\theta}{\theta^*}\right)+\xi^*\left(1-\tfrac{\theta^*}{\theta}\right)+\lambda^*\left(\tfrac{1}{\theta}-\tfrac{1}{\theta^*}\right) \\ & = \tfrac{1}{\theta^*}\int_0^{\theta^*} q(\tau)d\tau+\xi^* \ln\ \left(\tfrac{\theta}{\theta^*}\right). \end{aligned}$$

□

Proposition 3.1. *(Piecewise linear envelope) Suppose that* $\Pi^*(\theta)$ *is the profit function of a robust mechanism. Then,* $\Pi^*(\theta)=\max\{\xi^*\theta-\lambda^*,0\}$*, for some* $\xi^*>0$ *and* $\lambda^*\geq 0$ *(i.e., it is piecewise linear).*

Proof of Proposition 3.1. Let $m=(q,p)$ be any feasible mechanism and $\Pi(\theta)$ the associated profit function. By Lagrangian approach above, we know that there exist $\xi^*>0$ and $\lambda^*\geq 0$ such that $\Pi(\theta)\geq\xi^*\theta-\lambda^*$, for all θ. By Lemma 3.2, $q(\theta)\geq\frac{1}{\theta^*}\int_0^{\theta^*}q(\tau)d\tau+\xi^*\ln\ \left(\frac{\theta}{\theta^*}\right)$, for all θ, where $\theta^*=\lambda^*/\xi^*$. In particular, since $q(1)\leq 1$, we have that $\xi^*\ln\ \left(\frac{1}{\theta^*}\right)\leq 1$. Therefore, if $q^*(\theta)=\ \max\ \left\{\xi^*\ln\ \left(\frac{\theta}{\theta^*}\right),0\right\}$, then $q^*(1)\leq 1$ and

$$\Pi(\theta)\geq\Pi^*(\theta):=\theta q^*(\theta)-\int_0^\theta q^*(\tau)d\tau=\ \max\ \{\xi^*(\theta-\theta^*),0\},$$

i.e., the allocation $q^*(\cdot)$ is feasible allocation and attains the lower envelope profit, which proves the result. Finally, notice that the robust mechanism, the constraint $\xi^*\ln\ \left(\frac{1}{\theta^*}\right)\leq 1$ should bind, which implies that $\theta^*>0$, once $\xi^*>0$. Hence, $\lambda^*>0$. □

Proposition 3.1 shows that a robust mechanism imposes restrictions on the payoff levels. Only knowing the first moment of the distribution from which the consumer's valuation is drawn, the best the seller can do to insure against ambiguity is to design a mechanism that induces profits which are linear in the values conditional on sales. Under the interpretation that, ex-ante, the seller knows the consumer's expected willingness to pay, but the latter acquires additional information after the mechanism is offered, the convexity (or call-option format) of the profit function is the seller's optimal response to the added volatility in the consumer's valuation stemming from the information acquisition stage.

3.2 Optimal mechanism

Letting $F^*\in\mathcal{F}$ be a worst case distribution, from Proposition 3.1, the seller's expected profits $\Pi^*(\theta)$ at a robust mechanism is

$$\int\Pi^*(\theta)\,dF^*(\theta)=\int\max\{\xi^*\theta-\lambda^*,0\}\,dF^*(\theta)=\xi^*[k-\theta^*],$$

where $\theta^* = \lambda^*/\xi^*$ is the marginal consumer type – i.e., the consumer type such that the mechanism prescribes sales for all $\theta > \theta^*$. Moreover, at a robust mechanism $q^*(\cdot)$,

$$\Pi^*(\theta) \equiv \theta q^*(\theta) - \int_0^\theta q^*(\tau)\,d\tau = \xi^*\theta - \lambda^*, \text{ for all } \theta \in [\theta^*, 1].$$

Differentiating the above condition, we get

$$\theta \frac{dq^*}{d\theta}(\theta) = \xi^*, \text{for all } \theta \in [\theta^*, 1]$$

which implies that

$$q^*(\theta) = \xi^* \ln\left(\frac{\theta}{\theta^*}\right), \text{ for all } \theta \in [\theta^*, 1].$$

The monopolist's problem can then be simplified to

$$\max_{\xi^* \geq 0, \theta^* \in [0,1]} \xi^* [k - \theta^*] \tag{4}$$

subject to

$$\xi^* \ln\left(\frac{\theta}{\theta^*}\right) \leq 1 \text{ for all } \theta \in (0, 1]. \tag{5}$$

For the next proposition let us consider the (implicit) solution $\tilde{k} \in (0, k)$ of the following equation:[9]

$$\tilde{k}\left(1 - \ln \tilde{k}\right) = k. \tag{6}$$

Then, the solution to problem (4) yields:

Proposition 3.2. *(Robust mechanism) The optimal robust allocation is given by*

$$q^*(\theta) = \begin{cases} 0, & \text{if } \theta \leq \widetilde{k} \\ 1 - \frac{\ln \theta}{\ln \tilde{k}}, & \text{if } \theta \geq \widetilde{k} \end{cases}$$

and the robust profit is given by $\tilde{k}$.

Proof of Proposition 3.2. Notice that, at optimal solution, the constraint (5) of problem (4) must be binding exactly at $\theta = 1$, i.e., $\xi^* = -1/\ln \theta^*$. The problem then simplifies to

$$\max_{\theta^* \in [0,1]} \varphi(\theta^*),$$

[9]From the proof of Proposition 3.2, it is clear that $\tilde{k}$ exists and is unique.

where $\varphi(\theta^*) = \frac{\theta^* - k}{\ln \theta^*}$. The first order condition amounts to $\varphi'(\theta^*) = \frac{1}{\ln \theta^*} + \frac{k - \theta^*}{\theta^* (\ln \theta^*)^2} = 0$ which is equivalent

$$\theta^*(1 - \ln \theta^*) = k.$$

Since φ is strictly concave (φ' is strictly decreasing), this last equation has a unique solution, which we call $\tilde{k}$, and is the solution of our maximization problem. Now the expected profit is

$$\xi^*[k - \theta^*] = -\frac{1}{\ln \tilde{k}}[k - \tilde{k}] = \tilde{k},$$

which concludes the proof. □

While, as in standard Bayesian selling mechanisms, there are no distortions at the top ($q^*(1) = 1$), the robust mechanism entails sales with probability smaller than one for all valuations $\theta < 1$. Hence, the mechanism distorts the allocation at the intensive margin. Although coming from a different source (uncertainty, rather than the curvature of payoffs as in Bayesian settings), the reason for this distortion is to price discriminate different consumer types. Considering the worst-case scenario, the seller will find it desirable to sell to consumers with low valuations. If he was, however, to sell with probability one to them, the amount he would be able to charge from infra marginal consumers would be small. By selling with probability smaller than one to low valuation consumers, the seller can charge more from infra marginal buyers. Price discrimination is the way by which the seller simultaneously insures against uncertainty and charges high prices from infra marginal consumers.

Implementation

There are many ways to implement the allocation in Proposition 3.2. A first natural one is through a non-linear tariff. In fact, for

$$q^*(\theta) = \begin{cases} 0, & \text{if } \theta \leq \widetilde{k} \\ 1 - \frac{\ln(\theta)}{\ln(\widetilde{k})}, & \text{if } \theta \geq \widetilde{k} \end{cases}. \tag{7}$$

Let

$$p^*(\theta) = \theta q^*(\theta) - \int_0^\theta q^*(\tau)\, d\tau, \tag{8}$$

we can then make use of the taxation principle to implement the robust direct mechanism through an indirect mechanism $(q, P^*(q))$ with $P^*(q) = p^*(\theta)$ for $q = q^*(\theta)$.

A more interesting way – and that explicitly uses the fact that the consumer's payoff is linear – to implement the robust mechanism is, however, through a *distribution* of posted prices.

Indeed, notice that the direct mechanism calls for a buyer of type θ to be assigned assigned the good with probability $q^*(\theta)$. At any given price p, a buyer will buy if and only if $p \leq \theta$. Now, assume that the seller commits to posting a price $p \in \left[\widetilde{k}, 1\right]$ drawn from the cumulative distribution

$$q^*(p) \text{ for all } p \in \left[\widetilde{k}, 1\right],$$

with $q^*(\cdot)$ from equation (7). It is easy to see that, if prices are drawn from $q^*(p)$, the probability that a consumer of valuation θ buys is exactly $q^*(\theta)$. Hence, we have:

Proposition 3.3. *(Implementation) Committing to posting a price drawn from the distribution* $q^*(p)$*, for all* $p \in \left[\widetilde{k}, 1\right]$*, is a robust selling mechanism.*

3.3 Robust mechanism as a Nash equilibrium

An alternative to the approach that makes use of Lagrangian techniques is to recast, as Bergemann and Schlag (2008) do in their minimax pricing problem, the robust design problem in terms of a zero-sum game between the monopolist and an adversary nature.

In such game, the monopolist's von-Neumann-Morgenstern utility is $\Pi(\theta)$, whereas nature's is $-\Pi(\theta)$; the monopolist chooses incentive compatible mechanisms in $\mathcal{M}$ and nature selects distributions in $\mathcal{F}$.

As argued in Bergemann and Schlag (2008), if (m^*, F^*) is a Nash equilibrium of such game, then m^* is a robust mechanism and F^* is a worst case distribution. Consider the density function on $[0, 1]$:

$$f^*(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \widetilde{k}) \\ \frac{\widetilde{k}}{\theta^2}, & \text{if } \theta \in [\tilde{k}, 1] \end{cases}. \tag{9}$$

The following proposition shows that this density characterizes the absolutely continuous part of the distribution F^*:

Proposition 3.4. *(Nash equilibrium) Let* $m^* = (q^*, p^*)$ *be the mechanism characterized by (7) and (8) and the distribution* F^* *with absolutely continuous part described by (9) and singular part characterized by the Dirac measure at* $\theta = 1$ *with mass of* $\tilde{k}$*. Then,* (m^*, F^*) *is the unique Nash equilibrium of the zero-sum game played by the nature and the monopolist.*

Proof of Proposition 3.4. 1) Characterization. We start by guessing that the cumulative distribution F^* of the worst case measure has a density f^*, except possibly at $\theta = 1$ where it may have a mass point. Notice that, for any implementable allocation $q(\cdot)$, integration by parts yields

$$\int_0^1 \left[\theta q(\theta) - \int_0^\theta q(\tau)\,d\tau\right] f^*(\theta)\,d\theta =$$
$$\int_0^1 \left[\theta - \frac{F_-^*(1) - F^*(\theta)}{f^*(\theta)}\right] q(\theta) f^*(\theta)\,d\theta + q(1) - \int_0^1 q(\tau)\,d\tau.$$

Therefore,

$$\begin{aligned} & \int_0^1 \left[\theta q(\theta) - \int_0^\theta q(\tau)\,d\tau\right] dF^*(\theta) \\ = & \int_0^1 \left[\theta - \frac{F_-^*(1) - F^*(\theta)}{f^*(\theta)}\right] q(\theta) f^*(\theta)\,d\theta + \left(1 - F_-^*(1)\right)\left[q(1) - \int_0^1 q(\tau)\,d\tau\right] \\ = & \int_0^1 \left[\theta - \frac{1 - F^*(\theta)}{f^*(\theta)}\right] q(\theta) f^*(\theta)\,d\theta + \left(1 - F_-^*(1)\right) q(1). \end{aligned}$$

By making

$$\theta - \frac{1 - F^*(\theta)}{f^*(\theta)} = 0, \tag{10}$$

nature guarantees that the seller will be indifferent among any incentive compatible mechanism with $q(1) = 1$. In particular, the one in equation (7) is a best reply by the monopolist if equation (10) is satisfied. Solving (10) amounts to solving

$$\frac{d}{d\theta}\left[\theta F^*(\theta)\right] = 1,$$

one then has

$$F^*(\theta) = 1 - \frac{a}{\theta},$$

and $f^*(\theta) = \frac{a}{\theta^2}$, for all $\theta \in [a, 1]$, where $a = 1 - F_-^*(1)$. Now, we know that $F^* \in \mathcal{F}$ and hence

$$k = \int_0^1 \theta dF^*(\theta) = \int_a^1 \theta f^*(\theta)\,d\theta + \left(1 - F_-^*(1)\right),$$

which implies that $k = a \int_a^1 \frac{d\theta}{\theta} + a$, or $k = a(1 - \ln\, a)$, which implies that $a = \widetilde{k}$.

The discussion that precedes Proposition 3.1 and the derivation of the mechanism in (7)

Proposition 3.2 establishes that nature is indifferent among any distribution in $\mathcal{F}$; if the monopolist chooses the mechanism in equations (7) and (8). Hence, F^* is a nature's best response.

2) Uniqueness. Given the Nash equilibrium $((q^*, p^*), F^*)$ characterized above, for all $F \in \mathcal{F}$

$$\tilde{k} = \int \Pi^*(\theta) dF^*(\theta) \leq \int \Pi^*(\theta) dF(\theta),$$

and for all non-decreasing positive allocation $q(\cdot)$ such that $q(1) \leq 1$

$$\int \Pi(\theta) dF^*(\theta) = \tilde{k} \left(q(1) - \int_0^{\tilde{k}} q(\tau) d\tau \right) \leq \tilde{k}, \tag{11}$$

where $\Pi(\theta) = \theta q(\theta) - \int_0^\theta q(\tau) d\tau$.

Suppose that there exists another Nash equilibrium $((\bar{q}, \bar{p}), \bar{F})$. The mechanism q^* assures at least an expected profit of $\tilde{k}$ and therefore the mechanism $\bar{q}$ gives

$$\int \bar{\Pi}(\theta) d\bar{F}(\theta) \geq \tilde{k},$$

where $\bar{\Pi}(\theta) = \theta \bar{q}(\theta) - \int_0^\theta q(\tau) d\tau$. On the other hand, since the nature is minimizing the expected profit, it cannot attain a payoff lower than $\tilde{k}$ when deviating to distribution F^*. Then, using (11) we must have

$$\int \bar{\Pi}(\theta) dF^*(\theta) = \tilde{k} \left(\bar{q}(1) - \int_0^{\tilde{k}} \bar{q}(\tau) d\tau \right) = \tilde{k}.$$

We then necessary have $\bar{q}(1) = 1$ and $\bar{q}\left(\tilde{k}_-\right) = 0$. Hence, by continuity $\bar{\Pi}(\tilde{k}) = 0$ and, by Proposition 3.1, $\tilde{k} \leq \bar{k}$, where $\bar{I} = [\bar{k}, 1]$ contains the support of $\bar{F}$. This implies that $\bar{F}\left(\tilde{k}-\right) = 0$. If $x\left(1 - \bar{F}(x)\right) > \tilde{k}$, then the mechanism

$$q(\theta) = \begin{cases} 0 & \text{se } \theta \leq x \\ 1 & x < \theta \leq 1 \end{cases},$$

gives profit $\int \left(\theta q(\theta) - \int_0^\theta q(\tau) d\tau\right) d\bar{F}(\theta) = \int_x^1 x d\bar{F}(x) = x\left(1 - \bar{F}(x)\right) > \tilde{k}$. Hence, $x\left(1 - \bar{F}(x)\right) \leq \tilde{k}$, for all x. Now

$$k = \int_0^1 \left(1 - \bar{F}(x)\right) dx = \tilde{k} + \int_{\tilde{k}}^1 \left(1 - \bar{F}(x)\right) dx \leq \tilde{k} + \int_{\tilde{k}}^1 \frac{\tilde{k}}{x} dx = \tilde{k} + \tilde{k} \ln \left(\frac{1}{\tilde{k}}\right) = k.$$

Therefore, $x\left(1 - \bar{F}(x)\right) = \tilde{k}$, for all $x > \tilde{k}$, and hence $\bar{F} = F^*$.

From Proposition 3.1, there exist $\bar{\xi} > 0$ and $\bar{\lambda} \geq 0$ such that $\bar{\Pi}(\theta) \geq \bar{\xi}\left(\theta - \bar{k}\right)$, for all θ, where $\bar{k} = \frac{\bar{\lambda}}{\bar{\xi}}$. By Lemma 3.2, we have

$$\bar{q}(\theta) \geq \frac{\int_0^{\bar{k}} \bar{q}(\tau)\, d\tau}{\bar{k}} + \bar{\xi} \ln \left(\frac{\theta}{\bar{k}}\right), \text{ for all } \theta.$$

In particular, $1 \geq \bar{\xi} \ln \left(\frac{1}{\bar{k}}\right)$. Since,

$$\bar{\xi}\left(k - \bar{k}\right) \leq \frac{k - \bar{k}}{\ln\left(\frac{1}{\bar{k}}\right)} < \frac{k - \tilde{k}}{\ln\left(\frac{1}{\tilde{k}}\right)} = \tilde{k}$$

if $\bar{k} \neq \tilde{k}$ we conclude that $\bar{k} = \tilde{k}$. Hence, $\bar{q}(\theta) \geq q^*(\theta)$, for all θ. Let $dG = a\delta_{\tilde{k}} + (1-a)\,\delta_1$ where $a = \frac{1-k}{1-\tilde{k}}$ for $\int \theta dG(\theta) = k$ and δ_θ is the Dirac measure at θ. Then,

$$\begin{aligned}\int \bar{\Pi}(\theta)\, dG(\theta) &= (1-a)\left(\bar{q}(1) - \int_0^1 \bar{q}(x)\, dx\right) + a\tilde{k} = (1-a)\left(1 - \int_{\tilde{k}}^1 \bar{q}(x)\, dx\right) + a\tilde{k} \\ &\leq (1-a)\left(1 - \int_{\tilde{k}}^1 q^*(x)\, dx\right) + a\tilde{k} = \int \tilde{\Pi}(\theta)\, dH(\theta) = \tilde{k}.\end{aligned}$$

Therefore, $\bar{q}(x) = q^*(x)$ almost surely and then $\bar{q} = q^*$. □

3.4 Arbitrary moment condition

In this section we show that the characterization presented can be extended to more general moment conditions. The optimal pricing schedule provides a revenue guarantee that only depends on the known moment condition. This is achieved by use of an allocation rule that generates an ex-post profit function that is equal to an affine transformation of the moment function at the region of positive production. Formally, this condition can be described as a differential equation defining the robust mechanism. The characterization of the robust mechanism is done by construction of a Nash equilibrium of the zero-sum game between the seller and an adversarial nature, as discussed in Section 3.3.

Consider an arbitrary continuously differentiable function $\kappa : \mathbb{R} \to \mathbb{R}$ satisfying $\kappa'(\cdot) > 0$.[10] In this case, the set of feasible distribution be

$$\mathcal{F} = \left\{F \in \Delta[0,1]\,;\, \int \kappa(\theta)\, dF(\theta) = k\right\}, \tag{12}$$

for some fixe $k \in (\kappa(0), \kappa(1))$.

[10]The argument can easily be extended to the general case of bounded non-decreasing functions.

In what follows, we derive the robust mechanism by constructing a strategy profile of the zero sum game between nature and the monopolist then showing that it is a Nash equilibrium.

Distribution

Define distribution $F^{\underline{k}}$ as

$$F^{\underline{k}}(\theta) \equiv \begin{cases} 0, & \text{if } \theta < \underline{k}, \\ 1 - \frac{\underline{k}}{\theta}, & \text{if } \theta \in [\underline{k}, 1), \\ 1, & \text{if } \theta = 1. \end{cases}$$

Higher $\underline{k}$ leads to a first-order stochastic increase in distribution $F^{\underline{k}}$ and, hence, the function that $\underline{k}$ to $\int \kappa(\tau)\, dF^{\underline{k}}(\tau)$ is continuous and strictly increasing. Also, it converges to $f(i)$ when $\underline{k} \to i$, for $i \in \{0, 1\}$. By the intermediate value theorem, there is a unique point $\tilde{k} \in (0, 1)$ that satisfies

$$\int \kappa(s)\, dF^{\tilde{k}}(s) = k.$$

Let us denote $F^* = F^{\tilde{k}}$.

Mechanism

Define $\chi^a : \left[\tilde{k}, 1\right] \to \mathbb{R}$ as the following[11]

$$\chi^a(\theta) \equiv \begin{cases} 0, & \text{if } \theta < \tilde{k}, \\ 1 - a \int_\theta^1 \frac{\kappa'(\tau)}{\tau} d\tau, & \text{if } \theta \geq \tilde{k}. \end{cases}$$

Simple derivations imply that the profit of the seller under production χ^a is given by

$$\Pi^a(\theta) = \begin{cases} 0, & \text{if } \theta < \tilde{k} \\ \chi^a\left(\tilde{k}\right) + a\left[\kappa(\theta) - \kappa\left(\tilde{k}\right)\right], & \text{if } \theta \geq \tilde{k}. \end{cases}$$

Define $\tilde{a} \equiv \left[\int_{\tilde{k}}^1 \frac{\kappa'(\tau)}{\tau} d\tau\right]^{-1}$ and let $\Pi^* \equiv \Pi^{\tilde{a}}$, $q^* = \chi^{\tilde{a}}$, transfer p^* as

$$p^*(\theta) \equiv \theta q^*(\theta) - \int_0^\theta q^*(\tau)\, d\tau,$$

[11]The function χ^a solves the differential equation $\theta q'(\theta) = a\kappa'(\theta)$, with final condition $q(1) = 1$.

and mechanism $m^* = (q^*, p^*)$. This mechanism leads to a profit function that is continuous and satisfies

$$\Pi^*(\theta) = \tilde{a}\left[\kappa(\theta) - \kappa\left(\tilde{k}\right)\right].$$

The next two Lemmas provide a characterization of a Nash equilibrium of this game.

Lemma 3.3. *(Nature's problem) The robust distribution F^* solves the problem*

$$\min_{F\in\mathcal{F}} \int \Pi^*(\theta) dF(\theta).$$

Proof of Lemma 3.3. Notice that $\Pi^*(\theta) = \tilde{a}\left[\kappa(\theta) - \kappa\left(\tilde{k}\right)\right]$ if $\theta \geq \tilde{k}$ and $\Pi^*(\theta) > \tilde{a}\left[\kappa(\theta) - \kappa\left(\tilde{k}\right)\right]$ if $\theta < \tilde{k}$. Hence, it follows that

$$\int \Pi^*(\theta)\, dF(\theta) \geq \int \tilde{a}\left[f(\theta) - f\left(\tilde{k}\right)\right] dF(\theta) = \tilde{a}\left[k - \kappa\left(\tilde{k}\right)\right],$$

for any $F \in \mathcal{F}$. The first inequality holds as an equality if and only if $supp(F) \subseteq \left[\tilde{k}, 1\right]$.

The following lemma shows that m^* solves the seller's problem. ☐

Lemma 3.4. *(Seller's problem) The robust mechanism m^* solves the problem*

$$\max_{m=(q,p)\in\mathcal{M}} \int \Pi_m(\theta)\, dF^*(\theta),$$

subject to $p(\theta) = \Pi_m(\theta) \equiv \theta q(\theta) - \int_0^\theta q(\tau)\, d\tau$ and $q(\cdot)$ is non-decreasing.

Proof of Lemma 3.4. Notice that for any incentive compatible mechanism $m = (q, p) \in \mathcal{M}$

$$p(\theta) \equiv \theta q(\theta) - \int_0^\theta q(\tau)\, d\tau,$$

which, using the distribution F^*, leads to

$$\int p(\theta)\, dF^*(\theta) = (1 - F^*(1))\, q(1).$$

Any feasible mechanism with $q(1) = 1$ maximizes revenue. ☐

These two results lead immediately to the following result.

Proposition 3.5. *(Robust revenue - arbitrary moment) The mechanism $m^* = (q^*, p^*)$ solves the robust revenue maximization problem with moment condition $\mathbb{E}\left[\kappa\left(\tilde{\theta}\right)\right] = k$.*

4 One period and M goods

We now move to the case in which the monopolist can sell $M > 1$ goods. For the case in which $M > T = 1$, the seller has to design a mechanism $m = (\boldsymbol{q}, p) \in \mathcal{M}$ to maximize his worst-case expected profits under all cumulative probability distributions on $[0,1]^M$ with the vector of mean $\boldsymbol{k} = (k^1, ..., k^M) \in [0,1]^M$, i.e., the set of possible distributions is given by

$$\mathcal{F} \equiv \left\{ F \in \Delta\left([0,1]^M\right); \int \theta^i dF(\theta) = k^i, \text{ for } i = 1 \ldots, M \right\}.$$

This section has three main sets of results. For an arbitrary mechanism, we provide a characterization of the worst-case type distribution. We show that the profit function has to be jointly linear on all type dimensions for the subset of types that have positive probability under the worst-case distribution.

Second, we consider two relevant subsets of mechanisms. We calculate the revenue-mazimizing mechanism that satisfies (i) separability and (ii) full bundling of the multiple goods. The optimal separable pricing scheme uses the optimal one-dimensional pricing scheme characterized in Section 3, in each dimension. The optimal bundling allocation replicates the one-dimensional optimal pricing rule applied to the average valuation the agent has over all goods.

Finally, in the symmetric case, where the moment condition is equal on all dimensions, we provide a full characterization of the optimal revenue. We show that there are many optimal mechanisms. Interestingly, full separation and full bundling of the multiple dimensions are optimal. This means that both constrained maximization problems provide a solution to the original revenue problem considered.

The seller's problem reads

$$\max_{m \in \mathcal{M}} \min_{F \in \mathcal{F}} \int p(\theta)\, dF(\theta)$$

subject to participation constraint

$$\mathcal{U}_m(\theta) := \theta \cdot \boldsymbol{q}(\theta) - p(\theta) \geq 0,$$

and incentive compatibility constraint

$$\mathcal{U}_m(\theta) \geq \theta \cdot \boldsymbol{q}\left(\widehat{\theta}\right) - p\left(\widehat{\theta}\right),$$

for all $\theta, \widehat{\theta} \in [0,1]^M$. We refer to a mechanism satisfying these as a feasible mechanism.

It is standard to show that a mechanism $m = (\boldsymbol{q}, p)$ is incentive compatible if and only if

$$\nabla \mathcal{U}_m(\theta) = \boldsymbol{q}(\theta) \text{ for a.e. } \theta \in [0,1]^M \tag{13}$$

and

$$\mathcal{U}_m(\theta) \text{ is convex.}$$

Using equation (13), one can write incentive compatible payments made to the seller as the difference between total surplus and the buyer's utility:

$$p(\theta) = \theta \cdot \boldsymbol{q}(\theta) - \mathcal{U}_m(\theta) = \theta \cdot \nabla \mathcal{U}_m(\theta) - \mathcal{U}_m(\theta). \tag{14}$$

Therefore, each feasible mechanism $m = (\boldsymbol{q}, p)$ can be associated to a non-negative convex function rent $\mathcal{U}_m$ that satisfies (13). Reciprocally, given a non-negative convex function rent $\mathcal{U}_m$ that satisfies (13), then the mechanism $m = (\boldsymbol{q}, p)$ that satisfies (13) and (14) is feasible.

Plugging this is in the seller's objective we get

$$\max_{\{\mathcal{U}_m(\theta)\geq 0 \text{ and convex}\}} \min_{F\in\mathcal{F}} \int \underbrace{[\theta \cdot \nabla \mathcal{U}_m(\theta) - \mathcal{U}_m(\theta)]}_{\Pi_m(\theta)} dF(\theta).$$

4.1 Worst case payoffs

Once again, it is convenient to start by fixing an arbitrary incentive compatible profit function $\Pi(\theta) = \theta \cdot \nabla \mathcal{U}(\theta) - \mathcal{U}(\theta)$, where $\mathcal{U}(\theta)$ is a rent function associated to a feasible mechanism, i.e., $\mathcal{U}(\theta) \geq 0$ and convex. Consider then the worst case expected profits associated with $\Pi(\theta)$:

$$\min_{F\in\mathcal{F}} \int \Pi(\theta)\, dF(\theta). \tag{15}$$

As in Lemma 3.1, we show that a solution $F^* \in \mathcal{F}$ to problem (15) exists. Moreover, there are multipliers $\lambda^* \geq 0$ and $\xi^* \in \mathbb{R}_+^M$ so that

$$\min_{F\in\mathcal{D}} \int \Pi(\theta)\, dF(\theta) + \lambda^* \left(\int dF(\theta) - 1 \right) + \sum_{i=1}^{M} \xi^{*i} \left(k^i - \int \theta^i dF(\theta) \right) = \int \Pi(\theta)\, dF^*(\theta),$$

where $\mathcal{D}$ is the space of functions $F : [0,1]^M \to \mathbb{R}_+$ non-decreasing, right continuous and bounded function in each dimension.

Ignoring constant terms in the Lagrangian functional, a worst case distribution minimizes:

$$L(F;\lambda^{*},\xi^{*})=\int[\Pi(\theta)+\lambda^{*}-\xi^{*}\cdot\theta]\,dF(\theta).$$

Similar arguments to the ones used in Section 3.1 imply that a solution F^{*} of this problem only places positive weight on the set

$$I=\left\{\theta\in[0,1]^{M};\Pi(\theta)+\lambda^{*}-\xi^{*}\cdot\theta=0\right\}.$$

We used this fact to prove the following result, which a weaker M-good counterpart of Proposition 3.1 and has an analogous proof.

Proposition 4.1. *(Piecewise linear envelope) Suppose that* $\Pi(\theta)$ *is a profit function. Then,* $\Pi(\theta)$ *has a lower envelope piecewise linear function, i.e.,* $\Pi(\theta)\geq\max\{\xi^{*}\cdot\theta-\lambda^{*},0\}$ *with equality at the support of* F^{*}*, which is the solution of (15).*

Unlike the one good case, we cannot guarantee the robust profit function is piecewise linear. The next subsection derives the necessary conditions of optimality under the assumption the profit function is piecewise linear. In subsection 4.3, for the relevant case of symmetric moment conditions, we show that the optimal mechanism indeed leads to a piecewise linear profit function.

4.2 Optimal mechanisms

For now consider a feasible mechanism with ex-post profit function $\Pi(\theta)$ and let F^{*} be the solution to problem (15). One can use the necessary conditions implied by Proposition 4.1 to obtain a characterization of the optimal mechanism. The profit function is liner in the set of valuations inside the support of F^{*}, i.e.,

$$\Pi(\theta)=\xi.\theta-\lambda,\ \text{for any }\theta\in supp(F^{*}).\tag{16}$$

Considering any valuation in the interior of $supp(F^{*})$ and assuming differentiability we have that

$$\nabla\Pi(\theta)=\nabla\mathcal{U}(\theta)+\nabla^{2}\mathcal{U}(\theta)\cdot\theta-\nabla\mathcal{U}(\theta)=\xi.$$

Noticing that $\nabla\mathcal{U}(\theta)=\mathbf{q}(\theta)$, such expression can be rewritten as $\nabla\mathbf{q}(\theta)\cdot\theta=\xi$ or, in matrix terms,

$$\begin{pmatrix}\frac{\partial q_{1}(\theta)}{\partial\theta^{1}} & \cdots & \frac{\partial q_{1}(\theta)}{\partial\theta^{M}}\\ \vdots & \ddots & \vdots\\ \frac{\partial q_{M}(\theta)}{\partial\theta^{1}} & \cdots & \frac{\partial q_{M}(\theta)}{\partial\theta^{M}}\end{pmatrix}\cdot\begin{pmatrix}\theta^{1}\\ \vdots\\ \theta^{M}\end{pmatrix}=\begin{pmatrix}\xi^{1}\\ \vdots\\ \xi^{M}\end{pmatrix}.\tag{17}$$

Equation (17) is a system of partial differential equations that a robust mechanism must satisfy. In what follows, we describe two distinct sets of solutions to this system and later argue they are indeed optimal robust mechanism.

Fully separable sale mechanisms

Consider a class of mechanisms $m = (\boldsymbol{q}, p)$ satisfying the following full separability condition:

$$\frac{\partial q_i}{\partial \theta^j}(\theta) = 0, \; i = 1, ..., M, \; i \neq j. \tag{18}$$

Equation (17) then boils down to

$$\theta^i \frac{dq_i}{d\theta^i}(\theta^i) = \xi_i, i = 1, ..., M.$$

If the monopolist decides not to "link" in any form the sales of different goods, the best he can do is to replicate, for each good i, the mechanism in Proposition 3.2 for each moment $k^i \in (0, 1)$.

Proposition 4.2. *(Separability) The best mechanism $m = (\boldsymbol{q}, p)$ among those that satisfy (18) has*

$$q_i^*(\theta) = \begin{cases} 0, & \text{if } \theta^i \leq \widetilde{k}^i \\ 1 - \frac{ln\ \theta^i}{ln\ (\widetilde{k}^i)}, & \text{if } \theta^i \geq \widetilde{k}^i, \end{cases}$$

for $i = 1, ..., M$, where $\tilde{k}^i$ is the solution of (6) for $k = k^i$ and the price is defined by (13) and (14).

Proof of Proposition 4.2. Let $m = (\boldsymbol{q}, p)$ be a feasible mechanism and the associated non-negative convex function rent $\mathcal{U}_m$ that satisfies (13) such that $\boldsymbol{q}(\cdot) = (q_1(\cdot), ..., q_M(\cdot))$ satisfies the capacity constraints:

$$0 \leq q_i(\theta) \leq 1, i = 1, ..., M.$$

Because of (18) and using the envelope we have

$$\mathcal{U}_m(\theta) = \sum_{i=1}^{M} U_i(\theta^i)$$

where $U_i(\theta^i) = \int_0^{\theta_i} q_i(\tau, \theta^{-i}) d\tau$. Define the distribution $F^* = \times_{i=1}^{M} F_i^* \in \mathcal{F}$, where F_i^* is the robust distribution of the one good case with mean k^i characterized by Proposition 3.4 when

$k = k^i$. The expected of profit of mechanism m with respect to the distribution F^* is given

$$\int \Pi_m(\theta) dF^*(\theta) = \sum_{i=1}^{M} \int \Pi_i(\theta^i) dF_i^*(\theta^i),$$

where $\Pi_i(\theta^i) = \theta_i \bar{q}_i(\theta^i) - U_i(\theta^i)$, $\bar{q}_i(\theta_i) = \int q_i(\theta^i, \theta^{-i}) dF_{-i}^*(\theta^{-i})$ and $F_{-i}^*(\theta^{-i}) = \times_{j \neq i} F_j^*(\theta^j)$. this implies that $\Pi_i(\theta^i)$ is the one good profit function associated to the feasible one good allocation $q_i(\cdot, \theta^{-i})$. Since F_i^* is the robust distribution for the one good case with mean k^i, the maximum expected attained by the seller which we denote by $\tilde{k}^i$. Therefore,

$$\int \Pi_i(\theta^i) dF_i^*(\theta^i) \leq \tilde{k}^i,$$

for all $i = 1, .., M$, and

$$\int \Pi_m(\theta) dF^*(\theta) \leq \sum_{i=1}^{M} \tilde{k}^i.$$

This implies that the nature can limit the expected profit of the seller by $\sum_{i=1}^{M} \tilde{k}_i$. However, notice that this upper bound of the expected profit can be attained if the seller chooses the mechanism $m^* = (\boldsymbol{q}^*, p)$ where q_i^* and p are defined in the statement of the proposition. □

Bundled sale mechanisms

Rather than selling each good separately, the seller could bundle the goods. In fact, let $\widetilde{\theta} \in [0,1]^M$ be such that $\lambda = \xi \cdot \widetilde{\theta}$. It can be easily seen that the allocation

$$\boldsymbol{q}(\theta) = \begin{cases} 0, & \text{if } \xi \cdot \left(\theta - \widetilde{\theta}\right) < 0 \\ \ln\left(\frac{\xi \cdot \theta}{\xi \cdot \widetilde{\theta}}\right) \xi, & \text{if } \xi \cdot \left(\theta - \widetilde{\theta}\right) \geq 0 \end{cases} \tag{19}$$

is also a solution of (17) that entails bundling along the vector ξ and leads to the profit function $\Pi(\theta) = \max \{\xi.\theta - \lambda, 0\}$. Under this mechanism, the worst case expected profits are

$$\int \Pi(\theta) dF^*(\theta) = \int \xi \cdot \left(\theta - \widetilde{\theta}\right) dF^*(\theta) = \xi \cdot \left[\mathbf{k} - \widetilde{\theta}\right].$$

Therefore, the best among all such mechanism solves

$$\max_{\xi, \widetilde{\theta} \geq 0} \xi \cdot \left[\mathbf{k} - \widetilde{\theta}\right] \tag{20}$$

subject to

$$\ln\left(\frac{\xi\cdot\theta}{\xi\cdot\widetilde{\theta}}\right)\xi\leq\mathbf{1}\text{ for all }\theta,$$

where $\mathbf{1}=(1,...,1)\in\mathbb{R}^M$ is the vector formed by "1" in all entries.

The next proposition characterizes the optimal bundling solution. The interesting features of this solution is that it displays full bundling (i.e., selling in the direction of the vector $\mathbf{1}$, which corresponding the gradient of capacity vector) and the cutoff that determines the exclusion is related to a kind of average of the cutoff of one good case in each dimension. In the following proposition we use that following notation: given a vector $\boldsymbol{x}=(x^1,...,x^M)$, denote $\bar{\boldsymbol{x}}=\frac{1}{M}\sum_{i=1}^{M}x_i$.

Proposition 4.3. *(Full bundling) The solution to problem (20) entails full bundling:*

$$q_i(\theta)=\begin{cases}0, & \text{if }\bar{\theta}\leq\theta^*\\ \frac{ln\left(M\tilde{k}_M\right)-\ ln\ \bar{\theta}}{ln\ \tilde{k}_M}\mathbf{1}, & \text{if }\bar{\theta}>\theta^*\end{cases}$$

where $\theta^*=-\frac{M\tilde{k}_M}{ln\ \tilde{k}_M}$ *and* $\tilde{k}_M$ *is the unique solution of (6) for* $k=\bar{\boldsymbol{k}}$. *Moreover, the robust profit is given by* $M.\tilde{k}_M$.

Proof. Defining $\eta=\xi\cdot\mathbf{1}$ and $\theta^*=\xi\cdot\tilde{\theta}$, the problem (20) is then equivalent to

$$\begin{aligned}&\max_{\eta,\xi,\theta^*\geq 0}\ \xi\cdot\boldsymbol{k}-\theta^*\\ &s.t.\ \mathbf{1}-\left(ln\left(\eta\right)-ln\left(\theta^*\right)\right)\xi\geq 0\\ &\qquad\eta-\xi\cdot\mathbf{1}\geq 0.\end{aligned}$$

Let $a\in\mathbb{R}^M_+$ and $b\geq 0$ be the Lagrangian multipliers of the constraints. Notice that the Lagrangian is a concave function and, therefore, the following first-order conditions are necessary and sufficient for optimality:

$$\begin{cases}-\frac{\xi\cdot a}{\eta}+b=0\\ -1+\frac{\xi\cdot a}{\theta^*}=0\\ \boldsymbol{k}-\left(\ \ln\eta-\ \ln\theta^*\right)a-b\mathbf{1}\leq 0\end{cases}$$

and the usual slackness conditions.

From the first two equations we get $\theta^*=\xi\cdot a=b\eta$. Guessing (and then verifying) that the constraints and last first-order conditions are binding, we have that

$$\xi = \frac{1}{\ln \eta - \ln \theta^*}\mathbf{1} \text{ and } a = \frac{1}{\ln \eta - \ln \theta^*}(\boldsymbol{k} - b\mathbf{1}).$$

Hence,

$$\eta = \xi \cdot \mathbf{1} = \frac{M}{\ln \eta - \ln \theta^*} \text{ and } \theta^* = \xi \cdot a = \frac{\sum_{i=1}^M k^i - bM}{(\ln \eta - \ln \theta^*)^2} = b\eta = \frac{bM}{\ln \eta - \ln \theta^*}.$$

This implies that

$$\frac{\bar{\boldsymbol{k}}}{b} - 1 = \ln \eta - \ln \theta^* = \frac{M}{\eta}.$$

Hence, $\theta^* = \eta \exp[-M/\eta]$. However, $\theta^* = b\eta$ implies that $b = \exp[-M/\eta]$. Plugging this back into the previous expression, we get

$$b(1 - \ln b) = \bar{\boldsymbol{k}},$$

which gives a unique solution for b, which pins down the values of θ^* and ξ:

$$\theta^* = -\frac{Mb}{\ln b} \text{ and } \xi = -\frac{1}{\ln b}\mathbf{1}.$$

Notice that substituting these values in (19) we get the expression of the optimal allocation and the optimal profit is given by

$$\xi \cdot \boldsymbol{k} - \theta^* = -\frac{\boldsymbol{k} \cdot \mathbf{1}}{\ln b} + \frac{Mb}{\ln b} = M\frac{b - \bar{\boldsymbol{k}}}{\ln b} = Mb.$$

□

According to Proposition 4.3, the monopolist sells the same quantity of each of the M goods to the consumer. The amount sold depends, in turn, on the sum of her valuations, an aggregate measure of her willingness to pay.

4.3 Nash equilibrium in the symmetric case

While the mechanisms in Propositions 4.2 and 4.3 satisfy a set of necessary conditions implied by robustness, it still remains to verify whether they are indeed optimal for the monopolist. For that, once again, it will prove useful to study Nash equilibria of the zero sum game involving the monopolist and an adversary nature. The strategy space of the seller is the set of incentive compatible and individually rational direct mechanisms, while the strategy space of nature is distributions in the set $\mathcal{F}$. The payoff of the seller is his expected revenue, which is equal to minus nature's payoff.

In this subsection, we will consider the symmetric case, i.e., $k_i = k$, for all $i = 1, ..., M$. Consider the following distribution

$$F^*(\theta) \equiv G(\min_{1\le i\le M} \theta^i),$$

where

$$G(\theta) \equiv \begin{cases} 0, & \text{if } \theta \in \left[0, \tilde{k}\right), \\ 1 - \frac{\tilde{k}}{\theta}, & \text{if } \theta \in [\tilde{k}, 1) \\ 1 & \text{if } \theta = 1 \end{cases}$$

and $\tilde{k}$ is characterized by (6).

This distribution leads to perfectly correlated types. Notice, moreover, that the marginal distribution of each coordinate is equal to the distribution derived in Proposition 3.4, for the one dimensional case. Also define $m^* = (\mathbf{q}^*, p^*)$ as the following

$$p^*(\theta) = \sum_{i=1}^{M} \left[\theta_i q_i^*(\theta) - \int_0^{\theta^i} q_i^*\left([\theta, s]^i\right) ds \right], \tag{21}$$

where

$$q_i^*(\theta) \equiv \begin{cases} 1 - \frac{\ln \theta^i}{\ln \tilde{k}}, & \text{if } \theta^i \ge \tilde{k} \\ 0, & \text{if } \theta^i < \tilde{k}, \end{cases} \tag{22}$$

$$[\theta]^i \equiv \left(\theta^1, \ldots, \theta^i, 0, \ldots, 0\right),$$

and

$$[\theta, s]^i \equiv \left(\theta^1, \ldots, \theta^{i-1}, s, 0, \ldots, 0\right)$$

are specific truncations of the vector θ

In the following we will show that (m^*, F^*) is a Nash equilibrium of the zero sum game between nature and the seller.

Lemma 4.1. *(Distribution optimality) The optimal distribution F^* solves*

$$\min_{F\in\mathcal{F}} \int p^*(\theta)\, dF(\theta).$$

Proof of Proposition 4.1. Using (21) and (22) we have that

$$p^*(\theta) = -\sum_{i=1}^{M} \frac{\left(\theta^i - \tilde{k}\right)^+}{\ln \tilde{k}}.$$

Notice that, for any $\theta \in [0,1]^M$

$$p^*(\theta) \geq -\sum_{i=1}^{M} \frac{\theta_i - \tilde{k}}{\ln \tilde{k}},$$

and the above holds as an inequality if $\theta \notin \times_{i=1}^{M} \left[\tilde{k}, 1\right]$.

Also, for any $F \in \mathcal{F}$ we have that

$$\int p^*(\theta) dF(\theta) \geq -\int \sum_{i=1}^{M} \frac{\theta^i - \tilde{k}}{\ln \tilde{k}} dF(\theta) = -\sum_{i=1}^{M} \frac{k - \tilde{k}}{\ln \tilde{k}} = M\tilde{k}.$$

The first inequality holds as an equality if $supp(F) \subseteq \times_{i=1}^{M} \left[\tilde{k}, 1\right]$ and as a strict inequality otherwise. Finally, notice that $F^* \in \mathcal{F}$ and $supp(F^*) = \times_{i=1}^{M} \left[\tilde{k}, 1\right]$. □

We now show that, given distribution $F^* \in \mathcal{F}$, the seller finds it optimal to choose mechanism m^*.

Lemma 4.2. *(Mechanism optimality) The mechanism m^* solves the revenue maximization problem defined by F^**

$$m^* \in \arg\max_{m \in \mathcal{M}} \int p(\theta) dF^*(\theta).$$

Proof of Proposition 4.2. Notice that using incentive constraints relative to types the diagonal vectors $\theta^M := \theta.\mathbf{1}$, $\theta'^M := \theta'.\mathbf{1} \in [0,1]^M$, where θ and θ' are scalar now with some abuse of notation, we have that

$$\sum_i \theta q_i\left(\theta^M\right) - p\left(\theta^M\right) = -p\left(0^M\right) + \int_0^\theta q^A(s) ds,$$

where $q^A(s) \equiv \sum_i q_i\left(s^M\right)$. Hence, through standard arguments we can rewrite profits as

$$\int \left[\theta q^A(\theta) + p\left(0^M\right) - \int_0^\theta q^A(s) ds\right] dF(\theta) = \int q^A(\theta)\left[\theta - (1 - F(\theta))\right] dF(\theta) + p\left(0^M\right)$$

which is equal to

$$(1 - F(1)) q^A(1) + p\left(0^M\right).$$

Taking into account the constraint $q^A(1) \leq M$ and $p\left(0^M\right) \leq 0$, we have that the mechanism is indeed optimal. □

It follows from the above results that the mechanism described in Proposition 4.2 is, in fact, an optimal robust mechanism for the case of M goods. We now show that the mechanism in Proposition 4.3 attains the same worst case value for the monopolist as the mechanism in Proposition 4.2. We then have:

Proposition 4.4. *(Optimal full bundling) A mechanism that entails sales of all goods in the same proportion (full bundling) attains the same worst case expected profits for the monopolist as a mechanism that sells the goods in a fully separable fashion. Both mechanisms are optimal.*

Proof of Proposition 4.4. We only have to show that the expected value of the Nash equilibrium when the seller is restricted to bundling is the same under separability. Indeed, it is straightforward to check that the solution obtained by Proposition 4.3, as shown in the proof, is exactly $M\tilde{k}$. □

5 Multiple periods

In this section we study how robust the solution in Section 3 is to repeated interactions. There are two main reasons to consider the role of dynamics in our model. First, even if the seller has little information about the buyer's value, extra information might become available in future periods, leading to better revenue extraction. However, ambiguity on the side of the seller regarding this additional information makes the result non-trivial. In our dynamic model this is incorporated by looking at pricing rules that potentially depend on previous consumption behavior by the buyer.

Second, knowledge of average valuations is naturally connected to learning and information acquisition, as conditional expectations with increasing information sets follow a martingale. However, information acquisition is a dynamic phenomenon that should be discussed in a dynamic model.

The dynamic mechanism design literature on revenue maximization (Courty and Li (2000); Battaglini (2005); Pavan et al. (2015)) has described the optimal pricing schedule with repeated interactions and known type distributions. One common feature of these papers is dependence of the optimal dynamic pricing on fine details of the joint distribution.[12]

[12]Specially, Pavan et al. (2015) highlight the role of impulse-response functions as determinants of the optimal distortions in the optimal contract. The calculation of these objects requires knowledge of the entire joint distribution.

In contrast, our mail result is the irrelevance of dynamics in the presence of large ambiguity, i.e., repeated static optimal pricing is optimal. The intuition for our result is as follows. Optimal static pricing provides the best revenue guarantee that only depends on knowledge of the average valuation. By using the repetition of the static pricing rule, the seller completely separates the multiple periods and obtains a guarantee that only depends on the ex-ante average valuation of each period. The martingale property implies this revenue is the sum of the static revenue over multiple periods.

Certainly the seller cannot obtain more than the repeated static revenue as long as the case of permanent types is in the set of possible processes. For this process the optimal revenue without ambiguity is equal to the repetition of the static revenue.

A central part of our analysis deals with the key distinction between the case of multiple goods and multiple periods: the sequential revelation of information. We assume that the buyer observes in each period his realized valuation. A consequence of the sequential revelation of information is a failure of the revelation principle. For a given (indirect) mechanism, the optimal reporting strategy is dependent on the type process. Hence, the implied direct mechanism connected with a specific outcome is dependent on the type distribution chosen by the infimum operator. One methodological contribution of this paper is to present a way of dealing with dynamics in a world with ambiguity. The seller chooses from the larger set of indirect mechanisms and, in the face of large ambiguity, considers all possible type distribution as well the whole set of optimal reporting strategies induced by this distribution.

We now introduce the formal elements present in the dynamic analysis. Similarly to the static model, the small information held by the seller is described by a set of type distributions that are considered possible. Instead of treating each period independently, we consider type processes that have the martingale property. Let F^* denote the critical distribution, R_1^* denote the robust revenue level and $m_1^* = (q^*, p^*)$ denote the optimal selling mechanism in the static model with average condition $k \in (0,1)$. The set of possible type distributions is $\mathcal{F} \subseteq \Delta\left([0,1]^T\right)$ is assumed to have the following properties.

Assumption. The set $\mathcal{F} \subseteq \Delta\left([0,1]^T\right)$ satisfies:
(i) Martingale property: $F \in \mathcal{F} \Rightarrow \int \theta_{t+1} dF\left(\theta^T\right) = \theta_t$ for any t satisfying $T-1 \geq t \geq 1$.
(ii) Possibility of permanent types: $F^{*,T} \in \mathcal{F}$, where it is defined by $F^{*,T}\left(\theta^T\right) = F^*\left(\min_t \theta_t\right)$.

The first statement describes types that are derived from a learning process.[13] The second

[13]In our model the buyer is not completely informed about his valuation after the first period. Our model can incorporate the following scenario: the utility generated by consumption in each period $t \geq 1$ is $v + \varepsilon_t$, where $v \sim F$ and $\varepsilon_t \sim G_t\left(\mid v, \varepsilon_1, \ldots, \varepsilon_{t-1}\right)$ satisfying $\int v dF(v) = k$ and $\int s dG_t\left(s \mid v, \varepsilon_1, \ldots, \varepsilon_{t-1}\right) = 0$ for all $t \geq 1$ and $(v, \varepsilon_1, \ldots, \varepsilon_{t-1})$. In the beginning of each period t the buyer observes $\theta_t \equiv v + \varepsilon_t$. Ignorance of the joint distribution of θ^T is generated by ignorance of objects $\left(F, (G_t)_{t\geq 1}\right)$, and our optimal pricing result applies as

one only states that a specific constant types distribution is contained in the feasible set. The distribution $F^{*,T}$ has perfect correlation across periods, with valuations distributed according to F^*, the critical distribution in the static case. A special case of $\mathcal{F}$ is given by all martingale processes with the initial type distribution satisfying the static restriction $\mathbb{E}[\theta_1] = k$.

As discussed above, the analysis of the dynamic case cannot make use of the revelation principle. Hence, we need to explicitly consider the optimal reporting strategies induced by a given mechanism and a type distribution. These are defined below.

Definition 5.1. For a fixed distribution $F \in \mathcal{F}$ and mechanism $m \in \overline{\mathcal{M}}$, a reporting strategy $\sigma_0 \in \Sigma_m$ is optimal if

$$\mathbb{E}\left[\mathcal{U}_m\left(\sigma_0 \mid \tilde{\theta}^T\right)\right] \geq \max\left\{\mathbb{E}\left[\mathcal{U}_m\left(\sigma \mid \tilde{\theta}^T\right)\right], 0\right\}$$

for all $\sigma \in \Sigma_m$. The set of optimal strategies is denoted as $\Sigma_{m,F}$.

A direct mechanism is interim-incentive compatible, given a distribution $F \in \mathcal{F}$, if strategy $\sigma^{TT} \in \Sigma_{m,F}$. The set of interim incentive-compatible direct mechanisms is given by$\mathcal{M}_F^I$. We highlight once again that the set of optimal strategies potentially depends both on the mechanism in place and the type distribution F.

Now we are able to describe the decision problem faced by the seller. He decides what (potentially indirect) mechanism to offer to the agent. However, the seller does not know which actual type distribution $F \in \mathcal{F}$ he is facing, as well as what optimal reporting strategy $\sigma \in \Sigma_{m,F}$ is followed by the buyer. On the face of this uncertainty once again the seller tries to obtain the highest possible revenue guarantee. Formally:

$$R_T^* \equiv \sup_{m \in \overline{\mathcal{M}}} \inf_{\left(F \in \mathcal{F}, \sigma \in \Sigma_{m,F}\right)} \int \Pi_m\left(\sigma \mid \theta^T\right) dF\left(\theta^T\right), \tag{23}$$

where, for any indirect mechanism $m = (q_t, p_t)_{t \in T}$, the notation $m \circ \sigma$ denotes the direct mechanism $(\tilde{q}_t, \tilde{p}_t)_{t \in T}$ defined as $\left(\tilde{q}_t\left(\theta^T\right), \tilde{p}_t\left(\theta^T\right)\right) \equiv \left(q_t\left(\sigma^T\left(\theta^T\right)\right), p_t\left(\sigma^T\left(\theta^T\right)\right)\right)$ for all $\theta^T \in [0,1]^T$ and $t \geq 1$.

We start by describing the revenue guarantee that the seller has from choosing the repetition of the static optimal mechanism. This pricing rule guarantees at least the sum of the static

long as the learning process $\left(F^*, (\delta_0)_{t\geq 1}\right)$ where there is no learning is a possibility for the seller. The shock $(\varepsilon_t)_t$ determines the transitory effects that affect one's utility from consumption. For example, the satisfaction derived from watching a movie can be generated by the underlying taste for movies or from watching it in good company.

revenue period by period, which is equal to

$$R_T \equiv \frac{1-\delta^{T+1}}{1-\delta} R_1^*.$$

Let m_T^* denote the independent repetition of the static mechanism $m_1^* = (q^*, p^*)$, i.e., it is given by $\left(q_t^*\left(\theta^T\right), p_t^*\left(\theta^T\right)\right) = \left(q^*\left(\theta_t\right), p^*\left(\theta_t\right)\right)$ for all θ^T and $t \geq 1$. Independent pricing leads to incentive constraints that are completely separable over periods. As a consequence agents have incentives to truthfully report their types period by period for any type distribution. In fact, we show that the buyer loses a strictly positive amount by choosing a reporting strategy that leads to a different allocation than truth-telling. This separability leads to the following revenue guarantee.

Lemma 5.1. (Revenue guarantee) The mechanism m_T^* guarantees revenue R_T, i.e.,f

$$\inf_{\left(F\in\mathcal{F},\sigma\in\Sigma_{m,F}\right)} \int \Pi_{m_T^*}\left(\sigma \mid \theta^T\right) dF\left(\theta^T\right) = R_T.$$

Proof of Lemma 5.1. First consider any $F \in \mathcal{F}$. Any optimal reporting strategy $\sigma \in \Sigma_{m^*,F}$ has the property that

$$\int \mathbf{1}\left\{\theta^T \in [0,1]^T ; \left(q_t\left(\sigma_t\left(\theta^T\right)\right), p_t\left(\sigma_t\left(\theta^T\right)\right)\right) = \left(q_t\left(\theta^T\right), p_t\left(\theta^T\right)\right), \text{ for all } t \in T\right\} dF\left(\theta^T\right) = 1.$$

This expression states that any optimal reporting strategy is equivalent to the truth-full reporting strategy. This occurs because mechanism m^* is ex-post incentive compatible. This means that all agents have strict incentives to report truthfully, except when their types is in the exclusion region and they are indifferent among several announcements that lead to the same allocation. We start by showing this property.

Since in mechanism m^* the allocation in period t only depends on the announcements in that period, it is necessarily the case that for $\theta^T \in supp\left(F\right)$:

$$\sigma_t\left(\theta^T\right) \in \arg\max_{\theta'} \theta_t q^*\left(\theta'\right) - p^*\left(\theta'\right),$$

where

$$q^*\left(\theta'\right) \equiv \max\left\{1 - \frac{\ln \theta'}{\ln \tilde{k}}, 0\right\}, \tag{24}$$

$$p^*\left(\theta'\right) \equiv \max\left\{\frac{\theta' - \tilde{k}}{\ln \tilde{k}}, 0\right\},$$

where $\tilde{k}$ is the solution of (6). Since the objective function is strictly concave for $\theta' > \tilde{k}$ and constant for $\theta' \in \left[0, \tilde{k}\right]$:

$$\arg\max_{\theta'} \theta_t q^* (\theta') - p^* (\theta') = \begin{cases} \theta_t, & \text{if } \theta_t > \tilde{k}, \\ \left[0, \tilde{k}\right], & \text{if } \theta_t \in \left[0, \tilde{k}\right]. \end{cases}$$

This implies that $\left(q_t \left(\sigma_t \left(\theta^T\right)\right), p_t \left(\sigma_t \left(\theta^T\right)\right)\right) = \left(q_t \left(\theta^T\right), p_t \left(\theta^T\right)\right)$ with probability one.

As a consequence,

$$\inf_{F \in \mathcal{F}, \sigma \in \Sigma_{m^*,F}} \int \Pi_{m^*} \left(\sigma \mid \theta^T\right) dF \left(\theta^T\right) = \inf_{F \in \mathcal{F}} \int \Pi_{m^*} \left(\sigma^{TT} \mid \theta^T\right) dF \left(\theta^T\right).$$

But $\Pi_{m^*} \left(\sigma^{TT} \mid \theta^T\right)$ is given by $\sum_{t=1}^{T} \left[\delta^t \max\left\{\frac{\theta_t - \tilde{k}}{\ln \tilde{k}}, 0\right\}\right]$. As a consequence it follows that

$$\inf_{F \in \mathcal{F}} \int \Pi_{m^*} \left(\sigma^{TT} \mid \theta^T\right) dF \left(\theta^T\right) \geq \sum_{t=1}^{T} \left[\delta^t \max\left\{\frac{\mathbb{E}\left[\theta_t\right] - \tilde{k}}{\ln \tilde{k}}, 0\right\}\right] = R_T.$$

And this holds as an equality if the marginal distribution $marg_t F \subseteq \left[\tilde{k}, 1\right]$, which is true for distribution $F^{*,T} \in \mathcal{F}$. □

A direct implication of this lemma is that the optimal robust revenue is at least equal to this guarantee.

Corollary 5.1. *(Lower bound) The optimal robust revenue is at least equal to R_T, i.e., $R_T^* \geq R_T$.*

Proof of Corollary 5.1. By definition the following inequality holds

$$R_T^* \geq \inf_{F \in \mathcal{F}, \sigma \in \Sigma_{m^*,F}} \int \Pi_{m^* \circ \sigma} \left(\theta^T\right) dF \left(\theta^T\right) = R_T.$$

□

We now show that, in fact, the seller cannot improve upon the revenue guarantee described above. The argument is very simple: the distribution with permanent types following the static critical distribution F^* generates the revenue guarantee R_T as the optimal revenue with known type distribution. As a consequence, by considering the worst possible distribution, the seller can only obtain lower revenue. This gives us an upper bound on the minimax value $\bar{w}$.

Lemma 5.2. *(Upper bound) The (known) type distribution $F^{*,T}$ leads to optimal revenue R_T:*

$$R_T \equiv \sup_{m \in \mathcal{M}^I_{F^{*,T}}} \int \Pi_m \left(\theta^T\right) dF^{*,T} \left(\theta^T\right),$$

and hence the minimax value $\bar{w}$ *satisfies*

$$\bar{w} \equiv \inf_{F \in \mathcal{F}} \sup_{\left(m \in \overline{\mathcal{M}}, \sigma \in \Sigma_{m,F}\right)} \int \Pi_m \left(\sigma \mid \theta^T\right) dF\left(\theta^T\right) \leq R_T.$$

Proof of Lemma 5.2. The mechanism m_T^* is in $\mathcal{M}_{F^*,T}^I$ since it is ex-post incentive compatible. We briefly show that it solves the revenue maximization problem with known distribution $F^{*,T}$. Consider an arbitrary mechanism $m \in \mathcal{M}_F^I$ and define $\mathcal{U}_m^E(\theta_1) \equiv \mathbb{E}\left[\mathcal{U}_m\left(\tilde{\theta}^T\right) \mid \theta_1\right]$. Any incentive compatible mechanism $m = (q_t, p_t)_{t \in T}$ satisfies:

$$\mathcal{U}_m^E(\theta_1) = \sum_{t=1}^{T} \delta^{t-1}\left[q_t\left([\theta_1]^T\right)\theta_1 - p_t\left([\theta_1]^T\right)\right] = \max_{\theta' \in [0,1]} \sum_{t=1}^{T} \delta^{t-1}\left[q_t\left([\theta']^T\right)\theta_1 - p_t\left([\theta']^T\right)\right],$$

which implies that, using $Q(\theta) \equiv \sum_{t=1}^{T} \delta^{t-1} q_t\left([\theta]^T\right)$,

$$\mathcal{U}_m^E(\theta_1) = \mathcal{U}_m^E(0) + \int_0^{\theta_1} Q(s)\, ds.$$

Then expected revenue, according to F^{**}, satisfies

$$\begin{aligned}
\int \Pi_m\left(\theta^T\right) dF^{**}\left(\theta^T\right) &= \int \left[\theta Q(\theta) - \int_0^{\theta_1} Q(s)\, ds\right] dF^*(\theta) - \mathcal{U}_m^E(0) \\
&= \int_0^1 Q(\theta)\left[\theta - \frac{1 - F^*(\theta)}{f^*(\theta)}\right] d\theta + \left[F^*(1) - F^*(1_-)\right] Q(1) \\
&= \left[F^*(1) - F^*(1_-)\right] Q(1) \\
&\leq \left[F^*(1) - F^*(1_-)\right] \sum_{t \in T} \delta^{t-1},
\end{aligned}$$

where the third inequality uses the definition of F^* and the last inequality follows from resource constraints $q_t\left(\theta^T\right) \leq 1$. The last term in this sequence of inequalities is the expected revenue obtained by m_T^* since it always sells maximal quantity following an announcement $\theta_t = 1$. This concludes the first part of the statement.

The second part follows from

$$\begin{aligned}
\bar{w} &\equiv \inf_{F\in\mathcal{F}} \sup_{\left(m\in\overline{\mathcal{M}},\sigma\in\Sigma_{m,F}\right)} \int \Pi_m\left(\sigma \mid \theta^T\right) dF\left(\theta^T\right) \\
&\leq \sup_{\left(m\in\overline{\mathcal{M}},\sigma\in\Sigma_{m,F}\right)} \int \Pi_m\left(\sigma \mid \theta^T\right) dF^{*,T}\left(\theta^T\right) \\
&= \sup_{m\in\mathcal{M}} \int \Pi_m\left(\theta^T\right) dF^{*,T}\left(\theta^T\right) = R_T.
\end{aligned}$$

Where the first inequality uses $F^{*,T} \in \mathcal{F}$ and the second inequality is an application of the revelation principle. □

Concluding the argument, lemmas 5.1 and 5.2 present upper and lower bounds that coincide. Hence the optimal revenue is equal to this level.

Proposition 5.1. *(Optimal revenue) The optimal interim robust revenue, as defined in (23), is equal to R_T and the mechanism m_T^* achieves it.*

Proof of Proposition 5.1. Our proof consists of the following system of inequalities: $R_T \geq \bar{w} \geq R_T^* \geq R_T$.

Since the first and the third inequalities follow from lemmas 5.1 and 5.2, we only need to show the second inequality.

Consider any $F_0 \in \mathcal{F}$ then

$$\begin{aligned}
R_T^* = \sup_{m\in\overline{\mathcal{M}}}\left[\inf_{\left(F\in\mathcal{F},\sigma\in\Sigma_{m,F}\right)} \int \Pi_{m\circ\sigma}\left(\theta^T\right) dF\left(\theta^T\right)\right] &\leq \sup_{m\in\overline{\mathcal{M}}}\left[\inf_{\sigma\in\Sigma_{m,F_0}} \int \Pi_{m\circ\sigma}\left(\theta^T\right) dF_0\left(\theta^T\right)\right] \\
&\leq \sup_{m\in\overline{\mathcal{M}}}\left[\sup_{\sigma\in\Sigma_{m,F_0}} \int \Pi_{m\circ\sigma}\left(\theta^T\right) dF_0\left(\theta^T\right)\right] \\
&= \sup_{\left(m\in\overline{\mathcal{M}},\sigma\in\Sigma_{m,F_0}\right)} \int \Pi_{m\circ\sigma}\left(\theta^T\right) dF_0\left(\theta^T\right).
\end{aligned}$$

The first inequality follows from the removal of one degree of freedom in the infimum operator. The second inequality follows by substituting an infimum by a supremum. The final equality follows from the revelation principle. But now taking the infimum over all possible distributions $F_0 \in \mathcal{F}$ leads to the result. □

The above results are easily extended to the case of multiple goods, combining the arguments presented here with the ones in Section 4. For completeness we present here the general result, with an omitted proof for briefness.

Proposition 5.2. *The optimal dynamic, multidimensional, robust mechanism is the period by period repetition of the static mechanism, which can either entail sales in a fully separated fashion, or full bundling.*

6 Conclusion

We have considered a seller's problem in designing a worst-case mechanism when facing, for T periods, a privately informed buyer and having knowledge, at each period, of a single moment of the distribution from which consumer's multidimensional values are drawn. The results and their interpretations have been extensively discussed in the introduction and the main text, so we conclude with avenues for future research.

In the spirit of Carroll (2013), it would be nice to consider a robust design for more general priors. Dealing with multiple buyers would also be a natural extension of what we have done in this paper. The main difficulties in tackling such extensions are technical. Regarding the former, extending the zero-sum game approach that we use to verify optimality for the case of M goods and T periods is not straightforward. Even for the single good case with general priors (the case considered by Carroll (2013)), it is not quite clear how to compute Nash equilibria of the zero-sum game played by the seller and nature. The case in which there are more than one buyer is even more challenging. In fact, if we were to use Myerson's trick of substituting the incentive compatible representation of the consumer's payoff into the seller's objective, and proceed as we did in this paper, we would end up with a system of Partial Differential Equations (PDE), whose solution is hard to obtain. We could, instead, assume symmetric buyers and combine Myerson's trick with Border (1991)'s conditions to solve for the optimal reduced form robust auction. The difficulty is then to solve a single buyer robust selling mechanism adding the constraints implied by Border (1991). Although challenging, we hope future research addresses these questions.

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